WAVES

Wave Motion

Wave motion is a periodic disturbance in a medium that carries energy from one point to another. There are two major types of waves: longitudinal and transverse.

Longitudinal wave is such that the particles causing the wave vibrate along the direction of propagation of the wave. Examples include sound waves. Transverse wave is such that the particle causing the disturbance vibrate perpendicular to the direction of propagation of the wave. Examples are electromagnetic waves and waves on a string. A longitudinal wave propagates via compression and rarefaction. For this reason, longitudinal waves are also called compressional waves.

Fig. 1.1 shows the nomenclature of wave motion. The maximum displacement from the origin in called the **amplitude**, *A*, of the wave. The point of maximum displacement is called the crest while the point of maximum displacement in the other direction (negative) is called the trough. The wavelength, λ, is the distance (or its equivalent) between two consecutive troughs or two crests. That is, the distance between two identical points on the graph: two consecutive points where the graph crosses the *x*-axis from negative to positive, or from positive to negative. It is the distance over which the wave repeats.

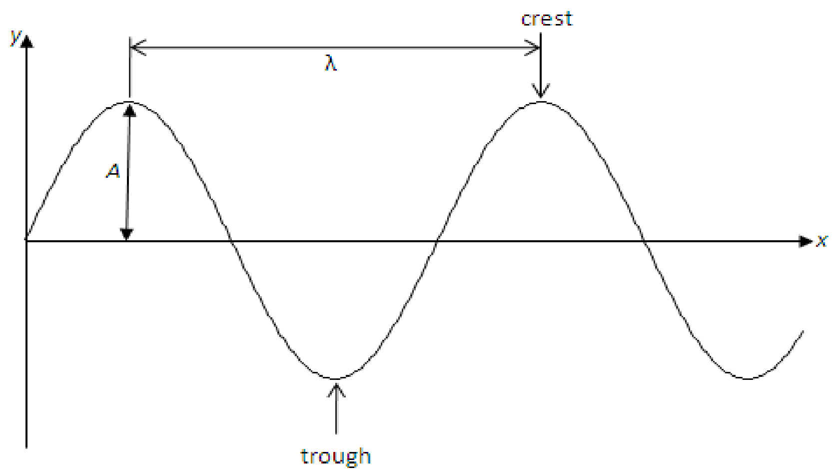

Fig. 1.1: Wave nomenclature with respect to distance

The wavelength of the wave is the ratio

$$k = \frac{2\pi}{\lambda} \qquad 1.1$$

and is the number of wavelengths per 2π unit of distance, usually measured in m^{-1}. You can think of this as analogous to distance = velocity$\times$time. A fixed distance of 2π is equal to a 'velocity' k and a fixed distance λ.

On the other hand, the wave could be such that the observer is at a fixed location and watches the wave pass by in time. Then, such an observer will see (in time) a crest, a trough pass by. Thus, he reckons wave motion with respect to time, instead of space. Then, Fig. 2 is relevant. The distance between two consecutive troughs is now the period, *T*, of the wave, the time over which the wave repeats.

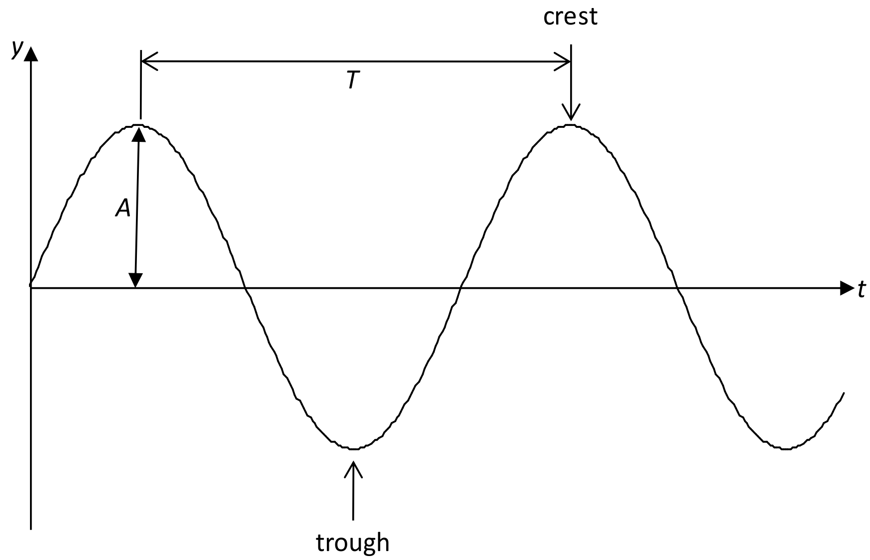

Fig. 1.2: Wave nomenclature with respect to time

The ratio,

$$\omega = \frac{2\pi}{T} \qquad 1.2$$

is called angular frequency of the wave, and since $\omega = 2\pi f$, the frequency $f = \frac{1}{T}$. The frequency is the number of waves that occurs in 1 second, measured in per second or more appropriately, Hertz, *Hz*.

Take a look at the graph of $y = A\sin\theta$. You will notice that *y* has the same value for equal values of θ or indeed $\theta + 2n\pi$, where n is an integer. We therefore say these values of θ and $\theta + 2n\pi$ are the same phase. You will then see that the argument of ***sine*** is the phase. If we write $z = A\sin(\omega t + \phi)$, the phase is $\omega t + \phi$.

We can write the wave more fully as

$$y = A\sin(kx - \omega t) \qquad 1.3$$

The phase is $kx - \omega t$. This is for a wave propagating in the positive *x*-direction. You can satisfy yourself that the velocity is positive (i.e., the motion is to the right), by setting $kx - \omega t$ equal to zero. Then

$$v = \frac{x}{t} = \frac{\omega}{k} \qquad 1.4$$

Both ω and k are positive. Hence, $v > 0$.

For a wave traveling to the left, a similar analysis will show that

$$v = \frac{x}{t} = -\frac{\omega}{k} \qquad 1.5$$

meaning that the velocity is negative.

v is called the **phase velocity** because that is the velocity with which a constant phase is propagated: $k\,x - \omega t = const$. Without loss of generality, we can set the constant equal to zero. Zero phase is as valid as any other one in view of our discussion so far.

There are several ways we can write equation 1.3:

$$y = A\sin(k\,x - \omega t) = A\sin(\frac{2\pi}{\lambda}x - 2\pi\ ft) = A\sin 2\pi\left(\frac{x}{\lambda} - f\ t\right) \qquad 1.6$$

From equation 1.6,

$$y = A\sin(k\,x - \omega t) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \qquad 1.7$$

Also from equation 1.6,

$$y = A\sin\frac{2\pi}{\lambda}\left(x - f\lambda\,t\right) \qquad 1.8$$

Since $v = f\lambda$,

$$y = A\sin\frac{2\pi}{\lambda}\left(x - v\,t\right) \qquad 1.9$$

Equation (1.9) could also have been realized by remembering that $v = \dfrac{\omega}{k}$ and applying this in equation 1.3:

$$y = A\sin k(x - \frac{\omega}{k}t) = A\sin k(x - vt) \qquad 1.10$$

Note that this is equivalent to equation 1.9: $k = \dfrac{2\pi}{\lambda}$.

Example

A wave is given as $y(x,t) = (8cm)\sin 2\pi\left(0.3\,x - 0.5t\right)$. Find its
(i) amplitude (b) frequency (c) period and (d) wavelength.

Solution

(a) The amplitude is $8\,cm$.

(b) From equation (1.6),

$$y(x,t) = (8cm)\sin 2\pi\left(0.3\,x - 0.5t\right) = A\sin 2\pi\left(\frac{x}{\lambda} - ft\right)$$

Hence,
$f = 0.5\,Hz$

(c) $T = \dfrac{1}{f} = \dfrac{1}{0.5} = 2s$

(d) $\lambda = \dfrac{10}{3}m$

We have assumed so far that the initial phase was zero. In the case where the initial phase is ϕ, the equations become:

$$y = A\sin(k\,x - \omega t + \phi) \qquad 1.11$$

Thus, at if we fix our attention at $x = 0$, in other words,

$$y = A\sin(-\omega t + \phi) \qquad 1.12$$

If we fix our attention at $t = 0$,

$$y = A\sin(kx + \phi) \qquad 1.13$$

From equation 1.11, if $\phi = 90^0$, then

$$y = A\sin(k\,x - \omega t + 90^0) = A\sin(\mu + 90^0) = A\cos(\mu) = A\cos(kx - \omega t) \qquad 1.14$$

Where $\mu = k\,x - \omega t$, and we have applied

$$\sin(90^0 + \theta) = \sin 90^0 \cos\theta + \sin\theta \cos 90^0 = \cos\theta$$

Speed of Waves on a String

Fig. 1.2a shows a wave pulse propagating over a small strip of an elastic string tightly stretched between two rigid supports. Let the tension in the string be **F** and the mass per unit length of the string μ kg/m. Of course, the string is a system of a large number of particles, but for our present purposes we will regard the mass as continuously distributed along the string. We assume that the amplitude of the wave pulse is very small (all compared to the length of the string). In this limit, the wave pulse produces a small perturbation in the tension and, we can assume to a good approximation, that the tension is the same everywhere along the string.

The resultant of the two approximately equal forces is as shown in Fig. 1.2b, and this is equal to the centripetal force acting on the element of length Δl of the string.

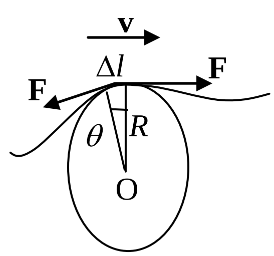

Fig. 1.2a

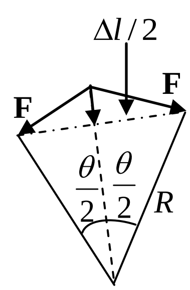

Fig. 1.2b

$$\mu\,\Delta l\,\frac{v^2}{R} = 2F\sin\frac{\theta}{2} \approx 2F \times \frac{\Delta l/2}{R} = F\,\frac{\Delta l}{R} \ \text{ (since } \theta \text{ is small)}$$

Therefore,

$$v = \sqrt{\frac{F}{\mu}}$$

Thus, the speed of the wave on a string in terms of the tension and the mass per unit length of the string. We observe that the speed is large if the tension is large and the mass per unit length is small. This is to be expected as a large tension can move a small mass very quickly.

Example

A long piece of piano wire of radius 0.3 mm is made of steel of density 7.7×10^3 kg/m^3. If the wire is under a tension of 1.1×10^3 N,

(a) (i) Calculate the speed of transverse waves on this wire?

(ii) What is the wavelength of a wave on this wire if its frequency is 260 Hz?

Solution

(i) We find the mass per unit length of the wire. Density is mass per unit volume, so that

volume = cross-sectional area $\times$ length

mass = volume $\times$ density = cross-sectional area $\times$ length $\times$ density

Hence,

$$\mu = \text{mass per unit length} = \text{cross-sectional area} \times \text{density}$$

$$= \pi r^2 \rho = 3.142 \times (0.3 \times 10^{-3} m)^2 \times 7.7 \times 10^3 kg/m^3$$

$$= 2.18 \times 10^{-3} kg/m$$

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{1.1 \times 10^3 N}{2.18 \times 10^{-3} kg/m}} = 7.1 \times 10^2 m/s$$

(ii) $$\lambda = \frac{v}{f} = \frac{7.1 \times 10^2 m/s}{260/s} = 2.73m$$

Example

2 m length of a certain rope has a mass 10 grams. A particle on the rope is described by the equation (x and y are in metres and t in seconds):

$$y(x,t) = 0.2\sin(x - 2t - 1.0)$$

(i) What is the tension in the rope?

(ii) Find the vertical position of a particle 0.1 m from the point $y(0,0)$ at time t = 0.5 s.

(iii) Calculate the period of the wave.

Solution

(i) The tension in the rope is given by,

$$v = \sqrt{\frac{F}{\mu}}$$

$$\mu = \frac{10 \times 10^{-3} kg}{2m} = 5 \times 10^{-3} kg/m, \; v = 20m/s$$

$$F = \mu v^2 = 5 \times 10^{-3} \times 2^2 = 0.2N$$

(ii) $$y(x,t) = 0.2\sin(x - 2t - 1.0) = 0.2\sin(0.1 - 2(0.5) - 1.0) = 0.2\sin(0.1) = 0.02m$$

Note that you work with radians. You would observe that

$$y(x,t) = A\sin(k\,x - \omega t + \phi) = A\sin k\left(\frac{k\,x}{k} - \frac{\omega}{k}t + \frac{\phi}{k}\right) = A\sin k\left(x - \frac{\omega}{k}t + \frac{\phi}{k}\right)$$

The unit of the argument of *sine* is still in radians. It just happens that in this case, k = 1 per m. In any case, unless otherwise stated, angles are measured in radians. Make sure your calculator is in 'radians' mode before you calculate $\sin(0.1)$. Do not forget to change back to degrees after you might be done with calculations in radians. Notice that ϕ / k has the unit of distance. This quantity will play a special role in the interference of waves. We see this later.

(iii) The period of the wave is given as ($k = 1m^{-1}$):

$$T = \frac{1}{f} = \frac{\lambda}{v} = \frac{2\pi / k}{v} = \frac{\left(\frac{2\pi}{1m^{-1}}\right)}{20m/s} = 0.1\pi\, s = 3.14\times 10^{-2}\, s$$

Particle Velocity, Slope of $y(x,t)$ with respect to x

Recall that Recall that

$$y(x,t) = A\sin(k\,x - \omega t + \phi)$$

We can differentiate $y(x,t)$ with respect to either x or t. Since it is a function of two variables, we do partial differentiation with respect to each of the two variables:

$$\frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t}[A\sin(k\,x - \omega t + \phi)] = A\frac{\partial}{\partial t}\sin(k\,x - \omega t + \phi) = -A\omega\cos(k\,x - \omega t + \phi)$$

$$\frac{\partial}{\partial x} y(x,t) = \frac{\partial}{\partial x}[A\sin(k\,x - \omega t + \phi)] = A\frac{\partial}{\partial x}\sin(k\,x - \omega t + \phi) = Ak\cos(k\,x - \omega t + \phi)$$

$\frac{\partial}{\partial t} y(x,t)$ is the particle velocity. This is different from the wave velocity, *v*. The particle velocity is the velocity say, of a particle of a rope on which there is a transverse wave. The particle moves up and down vertically while the wave travels down the rope. The wave propagates down the rope because particles transfer energy from one to the other without leaving their vertical positions. The vertical (we are assuming the wave is propagating in the horizontal direction) motion of the particles is about the equilibrium position. The particle velocity is in the vertical direction. In longitudinal waves, the particles vibrate along the direction of propagation of the wave about their equilibrium positions. We shall show that each particle performs simple harmonic motion about its equilibrium position.

$\frac{\partial}{\partial x} y(x,t)$ is the slope of the function $y(x,t)$ drawn as a function of position at the position *x*.

Example

Show that each particle in wave motion performs simple harmonic motion about its equilibrium position.

Solution

Particle velocity,

$$u = \frac{\partial}{\partial t} y(x,t) = A\omega \cos(k\,x - \omega t + \phi)$$

Particle acceleration,

$$\begin{aligned} a &= \frac{\partial^2}{\partial t^2} y(x,t) = \frac{\partial}{\partial t} u = A\omega \frac{\partial}{\partial t} \cos(k\,x - \omega t + \phi) \\ &= -A\omega^2 \sin(kx - \omega t + \phi) \\ &= -\omega^2 A \sin(k\,x - \omega t + \phi) \\ &= -\omega^2 x \end{aligned}$$

We recall that if acceleration obeys $a = -\omega^2 x$, then the body performs simple harmonic motion with angular frequency ω.

Interference of Waves

Let us superimpose two waves of equal frequency and amplitude traveling in the same direction.

$$y_1 = A \sin(k\,x - \omega t)$$
$$y_2 = A \sin(k\,x - \omega t)$$
$$y = y_1 + y_2 = 2A \sin(k\,x - \omega t)$$

There is nothing interesting here, as the composite wave only differs in amplitude.

Let us now make one of the waves to be ahead of the other by an initial phase ϕ.
Then,

$$y_1 = A \sin(k\,x - \omega t + \phi)$$
$$y_2 = A \sin(k\,x - \omega t)$$
$$\begin{aligned} y = y_1 + y_2 &= A \sin(k\,x - \omega t + \phi) + A \sin(k\,x - \omega t) \\ &= 2A \sin \frac{1}{2}(2k\,x - 2\omega t + \phi) \cos \frac{\phi}{2} \\ &= 2A \sin(k\,x - \omega t + \phi/2) \cos \frac{\phi}{2} \\ &= \left(2A \cos \frac{\phi}{2} \right) \sin(k\,x - \omega t + \phi/2) \end{aligned}$$

When $\phi = 0$, we are back to the trivial case. Fig. 3 shows the superimposition of two such waves with $\phi = \pi/6$. In Fig. 1.4, $\phi = \pi$, and the two waves are out of phase – the vibration is zero as the waves cancel out.

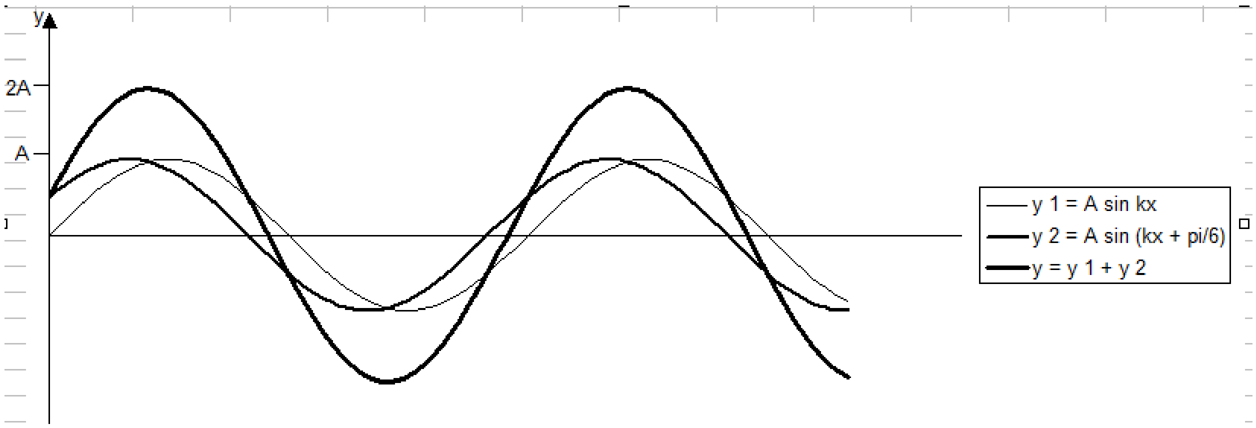

Fig. 1.3: Superposition of waves $y_1 = A\sin(kx + \pi/6)$ and $y_2 = A\sin kx$

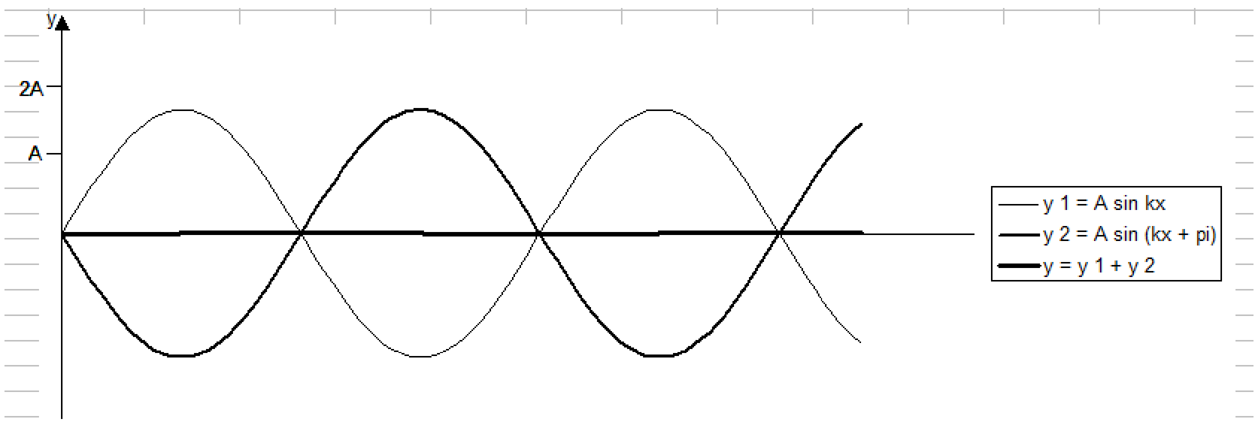

Fig. 1.4: Superposition of waves $y_1 = A\sin(kx + \pi)$ and $y_2 = A\sin kx$

Note that it is not only zero phase difference that gives constructive interference, but also $\phi = 2\pi$, 4π, etc., while $\phi = \pi$, 3π, etc., give destructive interference. The corresponding path differences are $\frac{\phi}{k}$. Recall that we said we shall be making use of this expression later. As such, constructive interference obtains when the path difference is,

$$0, \frac{2\pi}{k}, \frac{4\pi}{k}, \text{ etc., or } 0, \lambda, 2\lambda, \text{ etc.}$$

Destructive interference occurs when the path difference is,

$$\frac{\pi}{k}, \frac{3\pi}{k}, \text{ etc., or } \frac{\lambda}{2}, \frac{3}{2}\lambda, \frac{5}{2}\lambda, \text{ etc.}$$

Path differences corresponding to integral wavelengths interfere constructively while half-integral path differences give destructive interference.

Standing Waves

What happens when waves move on a string between two rigid supports? We expect a perfect reflection at the barriers. The reflected wave travels to the left and has the same angular frequency and wave number. Hence we can write

$$y_i = A\sin(kx - \omega t)$$

$$y_r = A\sin(kx + \omega t)$$
$$y = y_i + y_r = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$
$$= 2A\sin kx \cos \omega t$$

This is the equation of a standing wave. A progressive wave has a phase of the form $\omega t \mp kx + \phi$, from which it is apparent that the velocity of propagation is $\pm v$. If we fix x, then, $\cos kx$ is a constant, and we can then write equation … as

$$y(t) = B\cos \omega t$$

where $B = 2A\sin kx$. It is clear then that each particle performs simple harmonic motion as a function of time with angular frequency ω. But then, notice that, the amplitude B is a function just of *x*. If $x = 0$, for instance, B remains zero at all times. The same goes for the case $kx = n\pi$, where n = 0, 1, 2, …

$$kx = \pi, 2\pi, ...$$

or equivalently, $x = \dfrac{n\pi}{k} = \dfrac{n\pi}{2\pi/\lambda} = n\dfrac{\lambda}{2}$,

$$x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, ...$$

These are the nodes. The distance between them is $\lambda/2$.

There is another set of points, where $kx = (2n+1)\dfrac{\pi}{2}$, where n = 1, 2, 3, …, $B = 2A$ and the displacement from equilibrium is maximum. These are antinodes of the standing wave. They are located at *x*, such that

$$kx = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, ...$$

or equivalently,

$$x = \frac{n\pi}{2k} = \frac{n\pi}{2(2\pi/\lambda)} = \frac{n\lambda}{4},$$
$$x = \frac{\lambda}{4}, \frac{\lambda}{2}, \frac{3}{4}\lambda, ...$$

The distance between antinodes is $\lambda/2$.

The nodes are points that remain stationary throughout the course of the existence of a standing wave. As such, no energy could be propagated by a standing wave. Recall that in simple harmonic motion, there is energy interchange between the potential and the kinetic mode. Such is the case here as each particle performs simple harmonic motion, keeping its energy within its own ‘bank,’ with its mechanical energy kept within its own simple harmonic motion, hence the term ‘standing wave’ as the energies ‘stand.’

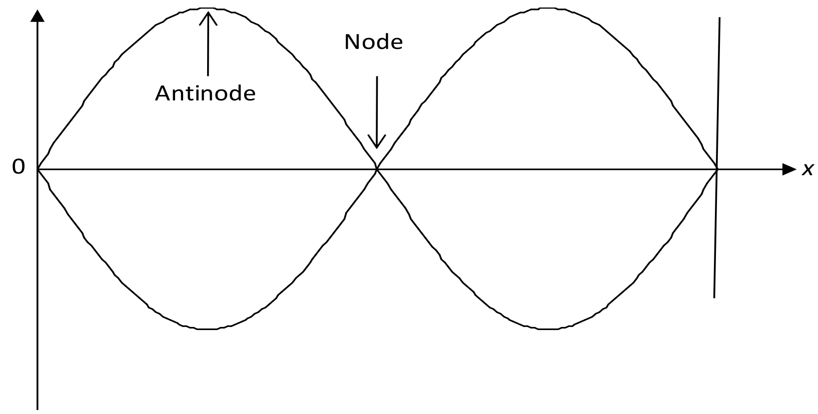

Fig. 1.5: Standing wave showing one wavelength between the rigid supports

Resonance

A body in vibration has its own natural frequency. When set in motion, a vibrating body could lose energy by interacting with its environment. For example, in the case of a string between two rigid supports, there could be energy loss due to imperfectly rigid supports, as well that due to air resistance. If an external force is applied vibration could be sustained. If we could vary the frequency of this force, then when the forcing frequency is equal to the natural frequency of the system, the amplitude of vibration becomes very large and the body is said to be in resonance. It is said to resonate with the applied force at that frequency.

From Fig. 1.5, we can see that the distance between two nodes is $\lambda/2$. Indeed, the number of half-wavelengths in any such string must be $\frac{n\lambda}{2}$, where n is an integer. Thus, we can say,

$$l = \frac{n\lambda}{2}$$

where l is the length of the string.

Rearranging,

$$\lambda = \frac{2l}{n}$$

$n = 1, 2, 3, \ldots$

But

$$v = f\lambda = f\,\frac{2l}{n}$$

Also,

$$v = \sqrt{\frac{F}{\mu}}$$

Equating equations () and (),

$$f = \frac{n}{2l}\sqrt{\frac{F}{\mu}}$$

These are the natural frequencies (of oscillation) of the system.

To set the string into resonance, the forcing frequency must have one of the values prescribed by equation ().

Example

A string is clamped at two ends and set into vibration such that the standing wave has five loops. A tuning fork of frequency 750 Hz sets the string into vibration. It is observed that the tension in the string is 2.5 N and the mass per unit length 0.001 N/m.

(a) If the amplitude of the wave is 1.8 mm, find the length of the string.

(b) Write the equation for the displacement of the string.

Solution

(a) $f = \frac{n}{2l}\sqrt{\frac{F}{\mu}}$

Hence,

$$l = \frac{n}{2f} = \frac{5}{2\times 750}\times\sqrt{\frac{2.5}{.001}} = 0.167\,m$$

(b) The equation for the displacement of the string is,

$y = 2A\cos kx \sin\omega t$

$\lambda = v/f = 50/750 = 0.067m$, since $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{2.5}{.001}} = 50\,m/s$

We could also get this by noting that if there are 5 loops, then since each loop is $\lambda/2$, there would be two and a half wavelengths within the 0.167 m length. $0.167m/2.5 = 0.067m$

The period, *T*, is the reciprocal of the frequency, and is thus,

$$T = \frac{1}{750\,s^{-1}} = 1.333\times 10^{-3}$$

$$y = 2A\cos\frac{2\pi}{\lambda}x\sin 2\pi f\,t$$

$$= 2(1.8\times 10^{-3})\cos\frac{2\pi}{0.067}x\sin 2\pi(750)t$$

$$= 3.6\times 10^{-3}\cos 93.78x\sin 4712.39t$$

x and y are in metres and t in seconds.

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1. A wave of frequency 300 Hz has velocity of 250 m/s.
   (a) Calculate the distance between two points 30^0 out of phase
   (b) The phase difference between two displacements at a certain point separated in time by 10^{-2} s.

2. The time taken by a point on a sinusoidal wave to travel from the maximum displacement to its lowest point is found to be 3s. Calculate the period, frequency and the speed of the wave, given that the distance between two points at zero displacement (at a particular instant of time) is 1.5 m.

3. The equation describing a transverse wave is given as $y = y_m \sin\left(\frac{x}{3} - \frac{t}{4} + \phi\right)$.

(a) Calculate the frequency and the wavelength of the wave.
(b) If the particle displacement at time $t = 0$ and $x = 0$ is one quarter of the amplitude, calculate the initial phase.

4. A vibrating string is described by the equation $y = 0.5\sin(2x - 3t)$, where x and y are measured in metres. Calculate the mass per unit length (linear density) of the string, given that the tension in the string is 0.4 N.

5. A wave on a string is described by

$$y = 2\cos(0.02x - 28t - 0.7)$$

where all distances are in metres and time in seconds.

(a) What is the equation of the wave to be added to the wave to obtain a standing wave?
(b) What is the distance between consecutive nodes and antinodes?
(c) How is this standing wave different from one obtained by reflecting $y = 2\cos(0.02x - 28t)$ from two rigid supports?

6. Vibrations from a 450 Hz tuning fork sets a string between two supports into vibration in such a way that string produces one loop, and the amplitude of the vibration is 0.5 mm. If the wave speed of the string is 250 m/s,

(a) Calculate the length of the string.
(b) Write the equation for the displacement of the string as a function of position and time.

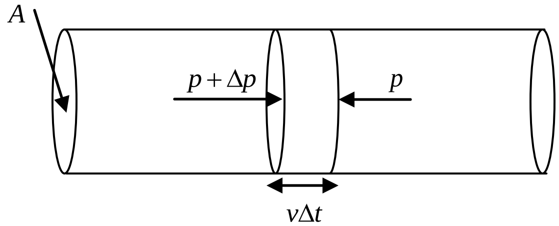
Sound Waves

Sound waves are an example of longitudinal mechanical waves. A mechanical wave is a wave that propagates as an oscillation of matter. This is why we say sound needs a material medium for its propagation. As we said before, individual particles performing simple harmonic motion do so about their equilibrium positions. As such, they do not travel far along the direction of propagation of energy.

As a sound wave moves from one point to another, particles of air perform simple harmonic motion, i.e., they vibrate back and forth in the same direction and the opposite direction of energy transport. The vibration (back and forth) of the particles in the direction of energy transport creates regions in which particles of the medium are pressed together and other regions where the particles are spread apart, compression and rarefaction.

Longitudinal mechanical waves have a wide range of frequencies, but our ears are sensitive only to those between 20 Hz and 20,000 Hz, the audible range. Longitudinal mechanical waves below 20 Hz are called infrasonic waves while those above 20,000 Hz are supersonic waves.

Sound waves are generated via compression and rarefaction. Consider a tube of length much greater than the width. Assume a piston is attached to the left end. Pushing the piston in compresses the fluid next to the piston. This fluid layer in turn passes this compression onto other fluid layers farther down the tube. The result is a compression pulse. If we now withdraw the piston, the fluid pressure and density in front of the piston drops, sending a rarefaction down the length of the tube. If the piston vibrates to and fro, then a continuous train of compression and rarefaction travels down the tube.



Applying Newton's laws to the fluid element when it is entering the compressional zone, we have

$$F = (p+\Delta p)A - pA = \Delta pA$$

where A is the cross-sectional area of the tube.

This is balanced by the inertial force $F = ma$. The mass of the fluid element is

$$\rho_0 \,(v\Delta t)A$$

and the acceleration

$$-\frac{\Delta v}{\Delta t}$$

Balancing these two forces,

$$\Delta pA = -\rho_0 (v\Delta t)A\frac{\Delta v}{\Delta t}$$

Therefore, $\rho_0 v = \dfrac{-\Delta p}{\Delta v}$,

Or $\rho_0 v^2 = -\dfrac{\Delta p}{\Delta v / v}$.

Let V be the volume of the fluid element before entering the compressional zone and ΔV be the change in volume when it is in the compressional zone. Then,

$$\frac{\Delta V}{V} = \frac{\Delta v}{v}$$

Hence, we conclude that

$$\rho_0 v^2 = -\frac{\Delta p}{\Delta V / V}$$

The term on the right is the bulk modulus of elasticity, B, of the fluid. Notice that $\Delta V / V$ is itself negative since it is a compression. Hence, B is positive. Thus, the velocity of the longitudinal pulse in the medium is

$$v = \sqrt{\frac{B}{\rho_0}}$$

The above analysis applies to pulses of any shape and to extended wave trains.

Thus, the more rigid a solid, the higher the bulk modulus and the faster the velocity of sound in it. The speed of sound in diamond is 12,000 m/s, in beryllium 12,890 m/s and in steel 5,900 m/s.

For a gaseous medium, we can express the Bulk modulus in terms of the undisturbed gas pressure p_0 as $B = \gamma p_0$, where γ is the ratio of specific heats (c_p / c_v) for the gas.

$$v = \sqrt{\frac{\gamma p_0}{\rho_0}}$$

For a solid thin rod, bulk modulus is replaced by the Young's modulus and for an extended solid, we have to make use of the shear modulus (a measure of elastic resistance to tangential or shearing force) and the bulk modulus.

Earthquakes propagate via the longitudinal P – wave and the transverse shear S – wave. S wave can only travel in a solid and as such cannot travel through the earth's core. The P wave is faster (on the average by 60%) than the S wave and the difference in their time of arrival can be used to determine the distance to the epicentre of the earthquake. The two waves travel in a more rigid medium.

The velocity of sound in some media:

Medium	Temperature (^0C)	Speed (m/s)
Air	0	330
Water	15	1450
Copper	20	3560

In an ideal gas,

$$v = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\frac{\gamma\, nRT}{V\rho_0}} = \sqrt{\frac{\gamma\, RT}{M}}$$

where M is the molecular mass of the gas.

$$v = \sqrt{\frac{\gamma\, RT}{M}} = \sqrt{\frac{273K \times \gamma\, RT}{273K \times M}} = \sqrt{\frac{273K \times \gamma\, R}{M}} \times \sqrt{\frac{T}{273K}}$$

$$v = \sqrt{\frac{273K \times 1.4 \times 8.31 J/mol\, K}{0.02897 kg/mol}} \times \sqrt{\frac{T}{273K}} = 331.11\sqrt{\frac{T}{273K}}$$

Therefore, for 273 K or 0 ^{0}C, the velocity of sound in air is 331.11 m/s.

We can write

$$v = \sqrt{\frac{\gamma\, RT}{M}} = \sqrt{\frac{1.4 \times 8.31}{0.02897}}\sqrt{T} = 20.04\sqrt{T} = 20.04\sqrt{\theta^0 C + 273}$$

Travelling Longitudinal Waves

We have treated the case where the particles involved in a vibration are perpendicular to the direction of propagation of the wave. In the case of sound waves, which are longitudinal waves, the direction of particle vibration is along the direction of the propagating wave. However, we note that each volume element oscillates about its mean position.

Thus, we can still write, for the wave propagating to the right,

$$y = y_m \sin(kx - \omega t)$$

But since sound waves are as a result of pressure variations, we may write the equation for variation of pressure. From

$$B = -\frac{\Delta p}{\Delta V / V},$$

we obtain

$$\Delta p = -\frac{B\Delta V}{V}$$

Suppose a layer of fluid has a thickness Δx and cross-sectional area A. The volume is $V = A\Delta x$. Let the volume change due to a pressure change be ΔV, then, $\Delta V = A\Delta y$, where Δy is the change in the thickness of the layer. Hence,

$$\frac{\Delta V}{V} = \frac{\Delta y}{\Delta x}$$

Consequently,

$$\Delta p = -\frac{B\Delta V}{V} = -\frac{B\Delta y}{\Delta x}$$

If we let $\Delta p = p$, the change from the undisturbed pressure p_0, then

$$p = -B\frac{\Delta y}{\Delta x}$$

In the limit $\Delta x \to 0$,

$$p = -B\frac{\partial y}{\partial x}.$$

Thus, if $y = y_m \sin(kx - \omega t)$, then,

$$p = -B\frac{\partial y}{\partial x} = -Bk\cos(kx - \omega t)$$

$$= -(\rho_0 v^2 k y_m)\cos(kx - \omega t)$$

$$= P\cos(kx - \omega t)$$

Beats

Let two sound waves be described by $y_1 = y_m \sin\omega_1 t$ and $y_2 = y_m \sin\omega_2 t$. Then,

$$y_1 + y_2 = 2y_m \sin(\frac{\omega_1 + \omega_2}{2})\cos(\frac{\omega_2 - \omega_1}{2})$$

Put $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$. Then,

$$y_1 + y_2 = 2y_m \sin 2\pi(\frac{f_1 + f_2}{2})\cos 2\pi(\frac{f_2 - f_1}{2})$$

Thus, the resulting vibration has a frequency $\bar{f} = \frac{1}{2}(f_1 + f_2)$. Then, there is an envelope,

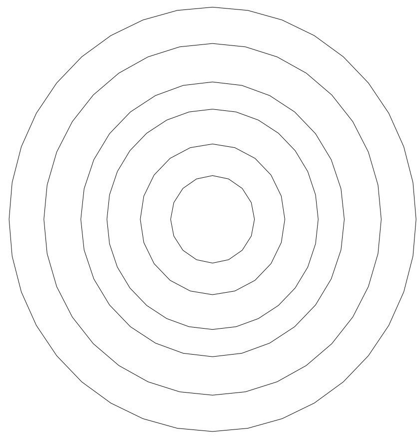
$$f_{env} = \frac{1}{2}(f_2 - f_1).$$

If $f_1 \approx f_2$, then f_{env} is very small, and the amplitude of the wave (the envelope) fluctuates slowly.

A beat, which is a maximum of amplitude, occurs when $\sin 2\pi(\frac{f_2 - f_1}{2})$ equals +1 or -1. Each of these occurs once in a cycle, and thus, the beat frequency is twice f_{env}, or equal to $f_2 - f_1$.

Doppler Effect

The frequency (or pitch) observed by an observer depends on the speed of the source and that of the observer itself. For example, a car's horn has a higher pitch when approaching a man (the observer) by the side of the road, than when it is receding.



Case 1: Source constant, observer moving towards it.

If observer had been stationary, his ears would have received vt/λ waves in time t, since the frequency is v/λ. But as he is moving with speed v_0, he receives an additional $v_0 t/\lambda$ waves in time t. Thus, observed frequency will be

$$f' = \frac{\frac{vt}{\lambda} + \frac{v_0 t}{\lambda}}{t} = \frac{v + v_0}{\lambda} = \frac{v + v_0}{v/f}$$

This can be rewritten as $f' = f + \frac{v_0}{v} f$

Thus, the observed frequency is the actual frequency emitted by the source plus an additional frequency $v_0 f / v$ due to the motion of the observer.

If the observer is moving away from the source (stationary in this case), there is a decrease in the frequency observed, a reduction of $v_0 f / v$, so that the observed frequency is

$$f' = f - \frac{v_0}{v} f$$

Thus, we conclude that $f' = f \pm \frac{v_0}{v} f$, depending on whether the observer is moving towards or away from the stationary source.

Consider the situation where the source is moving towards a stationary observer with velocity v_s. Since the frequency of the source is f, then during each vibration it travels a distance v_s / f. Each wavelength is shortened by this amount, meaning that the wavelength of the sound as observed by the observer is $\lambda' = \frac{v}{f} - \frac{v_s}{f}$. Consequently, the observer hears a sound of an increased frequency:

$$f' = \frac{v}{\lambda'} = \frac{v}{(\frac{v - v_s}{f})} = f\,\frac{v}{v - v_s}$$

For a source moving away from the observer, the observed frequency is decreased:

$$f' = \frac{v}{\lambda'} = \frac{v}{(\frac{v + v_s}{f})} = f\,\frac{v}{v + v_s}$$

Hence, we can write

$$f' = f\,\frac{v}{v \mp v_s}$$

depending on whether the source is moving towards or away from the observer.

For a situation where both source and observer are in motion,

$$f' = f\,\frac{v \pm v_o}{v \mp v_s}$$

The upper signs + and – correspond to the source and observer moving along the line joining the two in the direction towards the other, and the lower signs in the direction away from the other.

Examples

1. If a sound wave has a frequency of 380 Hz, find its wavelength in air, in water and in copper.

2. A man sees a lightning flash and then heard the thunder 5s later. Find the distance between him and the lightning.

3. Find the equation for the pressure variation of a travelling sound wave in air, if the particle displacement is maximum at time $t =$ 0s and $x =$ 0 m, and the displacement amplitude is 25 cm. Assume the undisturbed gas pressure is 10 Pa and the wavelength of the wave is 6.6 m.

4. Find the pressure amplitude of the resultant wave at a certain point in space, if the following waves are the cause of the pressure variation at the point.

$$p_1 = 5\sin\omega t$$

$$p_2 = 5\sin(\omega t - \frac{1}{4})$$

5. A sound wave of frequency 3000 Hz has a pressure of 12 Pa. Find its wavelength and amplitude of the particle speed.

6. Two identical piano wires kept under the same tension, have a fundamental frequency of 550 Hz. To achieve four beats per second when both vibrate simultaneously, find the fractional increase in the tension of one of the wires.

Examples

1. The pressure (in Pascals) in a travelling sound wave is given as

$$p = 2.4\sin\left(\frac{x}{2} - 150t\right)$$

where x is in metres and t in seconds. Calculate
(a) the frequency, (b) the wavelength, (c) the speed of the wave and
(b) the pressure amplitude.

2. The pressure variations at a certain point in space, given by two sound waves, are:

$$p_1 = A\sin \pi ft$$
$$p_2 = A\sin \pi(ft - \phi)$$

Find the pressure amplitude of the resultant wave at this point when $\phi = 0$, $\phi = \frac{1}{2}$, $\phi = \frac{1}{8}$.

3. Find the fractional increase in the tension of one piano wire that will lead to the occurrence of five beats per second if the two wires are identical and have a fundamental frequency of 550 Hz.

4. A siren emitting a sound of frequency 800 Hz moves towards an observer with a speed of 20 m/s.
(a) What is the frequency observed by the observer?
(b) What is the frequency of the sound the observer hears reflected off a cliff directly behind the siren?

5. An observer moving with speed 15 m/s approaches a source travelling at 10 m/s in the same direction.
(a) If the frequency of the source is 500 Hz, calculate the frequency recorded by the observer.
(b) If the source were to be travelling in the direction opposite that of the observer, what is the observed frequency?

Pitch, Loudness, Quality/Timbre, Intensity of sound, decibel.

The pitch of a sound wave is determined by its frequency. A female voice is usually at a higher pitch than the voice of a man. We say a woman's voice has a higher pitch and a man's voice a lower pitch. Soprano is at the highest pitch of all the parts sung in a choir.

Loudness is a factor of the amplitude of the wave. The Intensity of a sound wave is directly proportional to the square of its amplitude. If we double the amplitude of a wave, the sound is 4 times louder. The Bel expresses the logarithmic ratio between two levels of signal power, voltage, or current in electronics and communications. It is named after Alexander Graham Bell, who invented the telephone. The decibel (dB), one-tenth of a Bel, is more often used as the unit of measurement. This is because the smallest difference that can be detected by a human ear is a one-decibel difference in loudness between two sounds. It is worthy of note that the ear is more sensitive to sound of very small amplitude than it is to sound of very large amplitude. Thus, a nonlinear scale is appropriate. This is provided by the logarithmic measure

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

L is the loudness in decibels corresponding to a sound of intensity I in Watts per square metre and I_0 is the reference intensity. When $I = I_0$,

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)\text{dB}$$

The loudness at this intensity is zero decibel. I_0 is about $10^{-12} W/m^2$. Now, consider an intensity of 10^{-11} W/m^2.

$$L = 10\log_{10}\left(\frac{10^{-11}}{10^{-12}}\right) = 10\log_{10}10 = 10\text{ dB}$$

What about an increase from 10^{-1} to 1 W/m^2? This is an increase of 10 in the value of L.

$$L(10^{-1}W/m) = 10\log_{10}\left(\frac{10^{-1}}{10^{-12}}\right) = 10\log_{10}10^{11} = 110\text{ dB}$$

$$L(1W/m) = 10\log_{10}\left(\frac{1}{10^{-12}}\right) = 10\log_{10}10^{12} = 120\text{ dB}$$

You could also get this increase as

$$L(10^{-1} \to 1W/m) = 10\log_{10}\left(\frac{1}{10^{-1}}\right) = 10\log_{10}10 = 10\text{ dB}$$

Other units of sound intensity are, the neper (np) and the bel (B).

$$L = \frac{1}{2}\ln\left(\frac{I}{I_0}\right)np = \log_{10}\left(\frac{I}{I_0}\right)B = 10\log_{10}\left(\frac{I}{I_0}\right)dB$$

The decibel is a tenth of the bel. Notice that all these units are dimensionless.

1 bel = $\frac{1}{2}\ln 10$. I dB = $\frac{1}{20}\ln 10$

Example: Convert 10 decibel to neper.

$$1\text{ decibel} = \frac{1}{20}\frac{\ln\left(\frac{I}{I_0}\right)}{\log_{10}\left(\frac{I}{I_0}\right)} = \frac{1}{20}\times\frac{\log_e 10}{\log_{10}10} = \frac{1}{20}\times 2.3026 = 0.115\ np$$

10 decibel = 1.15 np = 1 bel

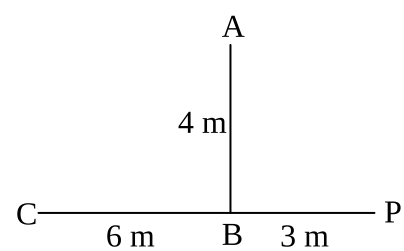
Variation of Intensity of Sound with distance from the source

The intensity of a sound wave falls with distance in an inverse-square manner. That is, $I \propto \frac{I_0}{r^2} \Rightarrow I = k\frac{I_0}{r^2}$. Very much like the behaviour of the electric field of a charge. This is because the charge is constant, just as the energy sent out at the source is constant at the source. The energy spreads out spherically very much as the electric field of a point charge or the gravitational field of a mass spread out. You must have come across this in the form of Gauss's law in electrostatics. Indeed, Sound intensity or acoustic intensity, is the power carried by sound waves per unit area in a direction perpendicular to the area.

Problem

A sound source is at P in the diagram. Calculate the ratio of the sound intensity at A, B and C.

$$I_A = k\frac{I_0}{5^2}, I_B = k\frac{I_0}{3^2}, I_C = k\frac{I_0}{9^2} \Rightarrow I_A : I_B : I_C = k\frac{I_0}{5^2} : k\frac{I_0}{3^2} : k\frac{I_0}{9^2} = \frac{1}{25} : \frac{1}{9} : \frac{1}{81} = 729 : 2025 : 225$$



Quality/Timbre

This depends on the number of overtones present in the sound and their relative intensities. This enables you to differentiate between two musical instruments having the same fundamental frequency. For instance, a tube opened at one end has only odd harmonics while a tube opened at both ends has both odd and even harmonics. The waveform of the latter is more complex than that of the former.