Periodic Motion

Any motion that is regular and repeating is referred to as a periodic motion. Most objects that vibrate do so in a regular and repeated manner; their vibrations are periodic. Periodic motion is common at all length scales in the universe. On the atomic (microscopic) level, atoms and molecules vibrate in solids. At the human level, macroscopic objects, such as guitar strings, vibrate. On the cosmological level, the universe itself has a repeating pattern of expansion and contraction. Examples of periodic motion include: (i) the motion of a mass on a spring; (ii) the motion of the clock pendulum; (iii) the rotation of the Earth about its axis; (iv) the Earth orbiting the Sun; (v) the motion of the Moon about the Earth; (vi) heartbeat; (vii) breathing; (viii) the oscillations of atoms in a molecule; (ix) rotating phonograph turntable (x) movement of a rocking chair etc. In all these cases, the object's motion "approximately" keeps repeating itself. In addition to the examples listed above, periodic motion occurs in some types of wave motion. The time required for each repetition is the *period* T and the number of repetitions made in a unit of time e.g. second, is the *frequency* f.

Periodic Motion in One Dimension

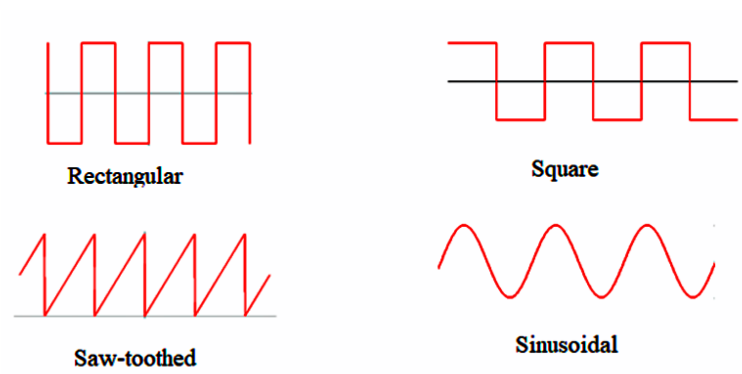
In one dimension, an object will perform periodic motion along the x- or y-axis. Suppose that the motion of an object occurs along the x-axis, the position of the object can be represented by its position function x(t). If the motion is periodic, it means that after an interval of period T:

$$x(t+T) = x(t) \tag{1a}$$

$$x(t+nT) = x(t) \quad n = 1, 2, 3, \ldots \tag{1b}$$

where T is the period of the motion.

The path or pattern of periodic motion may be linear, circular, elliptical or any other curve. The function $x(t)$ can have any shape e.g. rectangular wave, square wave, saw-toothed, sine/cosine wave etc., as long as the condition $x(t+T) = x(t)$ is satisfied.



In some cases, x(t) has a simple oscillating shape, i.e. simple sinusoidal form. If the motion is perfectly sinusoidal (sine or cosine function), then the motion is a "simple harmonic motion" (SHM).

Simple Harmonic Motion

A very common type of periodic motion is the simple harmonic motion (SHM). In SHM, the displacement, velocity and acceleration of the oscillating object can all be represented as sinusoidal functions of time. SHM arises whenever an object is made to oscillate about a position of equilibrium by a force that has the following characteristics: (i) the force is always directed towards the equilibrium point; (ii) the force has a magnitude which is proportional to the displacement from the equilibrium point.

In practice, when most stable systems are displaced slightly from equilibrium and then released, the periodic motion that follows can be treated as SHM, a combination of SHMs, or at least an approximation to one or more SHMs. For this reason, SHM can be regarded as a basic building block of far more complicated periodic motions. Vibrations and oscillations are part of our everyday life. They occur in a wide variety of forms: the buzzling of the alarm; the bounce of bed; the oscillations of loudspeaker, which in turn are produced by oscillations of charges in electric circuits; the vibrations of an electric razor, etc. Some are very welcome and aesthetically pleasing, such as in musical instruments. Others, such as the vibrations caused by machinery and traffic, are noisy and disturbing.

Conditions for Simple Harmonic Motion

Two conditions are necessary for SHM to occur: (i) An object must be displaced from its position of stable equilibrium and then released, and (ii) the restoring force must be directly proportional to the object's displacement from its equilibrium position. Three parameters are used to describe any single sinusoidal motion: the period (T), the amplitude (A), and the phase (δ or ϕ), a parameter that is related to the initial position at time t = 0. The amplitude of the motion is the maximum displacement from equilibrium position.

Kinematics of Simple Harmonic Motion

Suppose that an object in SHM starts off initially at $x = 0$, has some initial speed in the positive x-direction and an amplitude A. In this case, the motion of the object can be described by:

$$x(t) = A\sin\left(\frac{2\pi t}{T}\right) \qquad 2$$

T is the period of the motion and from circular motion $\omega = \frac{2\pi}{T} = 2\pi f$.

$\therefore x(t) = A\sin\omega t$

At time $t = 0$, the position is at $x = 0$ since $\sin(0) = 0$. The velocity of the particle at any time t is found by differentiating $x(t)$:

$$v(t) = \frac{dx}{dt} = \omega A\cos\omega t \qquad 3$$

The initial velocity is just $v(0) = \omega A$, and is in the $+x$ direction. ωA is the maximum speed of the object.

Generalized Mathematical Expressions for SHM

Equation (2) applies only to SHM in which the object starts at $x = 0$. For an object that starts off at $x = x_0$, where $x_0 \neq 0$, we need to add a phase shift, δ (or ϕ) measured in radians, to the argument of the sine function:

$$x(t) = A\sin(\omega t + \delta) \qquad 4$$

If $\delta > 0$ then the sine function is shifted to the right. If $\delta < 0$ then the sine function is shifted to the left. By adjusting the phase angle, we can shift the sine function to match the initial condition.

If the initial position is defined as $x(0) = x_0$, then

$x_0 = A\sin\delta$

$$\delta = \sin^{-1}\left(\frac{x_0}{A}\right) \tag{5}$$

The quantity δ is called the phase constant (shift or difference). It is determined by the initial conditions of the motion. It is the difference in the positions or states of two oscillating objects at a given time. Another representation of SHM is given as $x(t) = A\cos(\omega t + \delta)$. If at t = 0 the object has its maximum displacement in the positive x-direction, then δ = 0. if it has its maximum displacement in the negative x-direction, then δ = π. If at t = 0 the particle is moving through its equilibrium position with maximum velocity in the negative x-direction then δ = π/2. The quantity in the bracket $\omega t + \delta$ is called the phase.

Alternative Mathematical Expressions for SHM

The function $x(t) = A\sin(\omega t + \delta)$ can be expressed in an alternative form:

(i) Expand and substitute $\sin(\omega t + \delta) = \sin\omega t\cos\delta + \cos\omega t\sin\delta$ into Equation for $x(t)$

$x(t) = A\cos\delta\sin\omega t + A\sin\delta\cos\omega t$

(ii) Use the condition: At t = 0, $x_0 = x(0) = A\sin\delta$

$x(t) = A\cos\delta\sin\omega t + x_0\cos\omega t$

(iii) Differentiate x(t) to get: $v(t) = \omega A\cos(\omega t + \delta)$

(iv) Use the condition $v_0 = v(0) = \omega A\cos\delta$ to get $A\cos\delta = v_0/\omega$ and substituting into the equation for $x(t)$.

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t \tag{6}$$

Squaring $x_0 = A\sin\delta$ and $\frac{v_0}{\omega} = A\cos\delta$, then adding the two, we have

$$x_0^2 + \frac{v_0^2}{\omega^2} = A^2\left(\sin^2\delta + \cos^2\delta\right)$$

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$$

Also, dividing $x_0 = A\sin\delta$ by $\frac{v_0}{\omega} = A\cos\delta$, we have

$$\delta = \tan^{-1}\left(\frac{x_0}{v_0/\omega}\right)$$

We can also start with $x(t) = A\cos(\omega t + \delta)$, expanding it, we have

$x(t) = A\cos\delta\cos\omega t - A\sin\delta\sin\omega t$

At t = 0, $x_0 = x(0) = A\cos\delta$

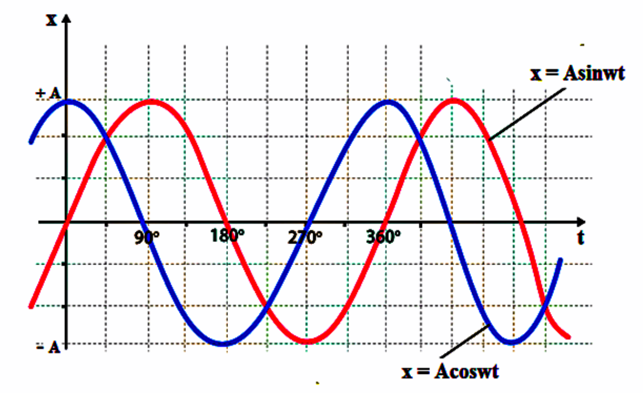
$x(t) = x_0\cos\omega t - A\sin\delta\sin\omega t$

Differentiating $x(t) = A\cos(\omega t + \delta)$, gives $x(t) = -\omega A\sin(\omega t + \delta)$. At t = 0, $v_0 = v(0) = -\omega A\sin\delta$ to get $A\sin\delta = -v_0/\omega$

Therefore, $x(t) = x_0\cos\omega t - \left(-\frac{v_0}{\omega}\sin\omega t\right) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$

$A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}}$ and $\delta = \tan^{-1}\left(-\dfrac{v_0/\omega}{x_0}\right)$

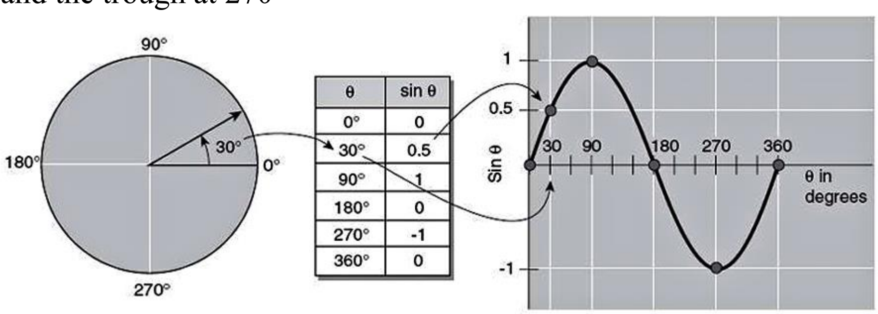
It is clear that both sine and cosine functions produce the same expression for the amplitude A, but the expressions for the phases δ are different. Both sine and cosine function represent SHM, but the initial conditions (without phase) are not the same. For sine form of SHM, at $t = 0$, $x = 0$ or $x_0 = 0$. For cosine form, at $t = 0$, $x_0 = A$. We can also have a situation where $x_0 \neq 0$ or $x_0 \neq A$.

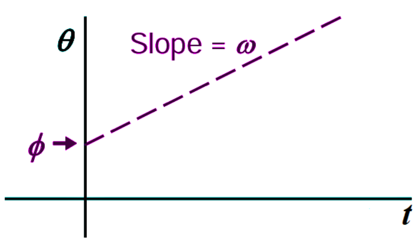


Phase and Phase Shift

The stage of the cycle of an oscillation is called its phase. Crest, trough, and zero can be used to describe the phases corresponding to maximum, minimum, and equilibrium, respectively. Because oscillation is described mathematically by a sine or cosine function, the phase is described by an angle θ. The unit for angle is the radian, where 2π radians = 360°.

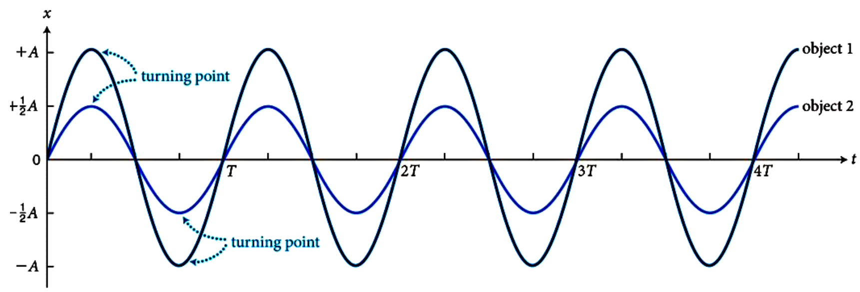
The figure below shows a graph sin θ on the y axis vs. the phase angle θ on the x axis. The crest is at 90° and the trough at 270°





ϕ is the value of the angle θ at $t = 0$. ω is the rate at which angle θ is increasing. Recall that the equation for a straight line is: $y = mx + c$, therefore, $\theta = \omega t + \phi$.

Phase shift or difference refers to the difference in the positions or states of two oscillating objects at any given point in time. It shows how much one oscillating object is ahead or behind another in terms of position or state. For example, two pendula swinging back and forth are said to be in phase if they both start swinging at the same time and at the same speed. However, if one starts swinging a bit later than the other, there will be a phase difference between them. Phase difference can be calculated by using the formula: $\Delta\phi = 2\pi\Delta t/T$, where $\Delta\phi$ is the phase difference, Δt is the time difference between the two oscillations, and T is the period of the oscillation. This formula is obtained based on the fact that one complete oscillation corresponds to a phase change of 2π radians. In the study of waves, the phase difference between two points on a wave can give information on the wave's speed and direction. Similarly, in sound, the phase difference between two sound waves gives the information about the sound's pitch and volume.



Shifting Trigonometrical Functions

Trigonometrical functions can be shifted to match the initial condition.

$$x = A\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\omega t \pm \delta)$$

The minus sign means that the phase δ is shifted to the right. A plus sign indicates the phase is shifted to the left.

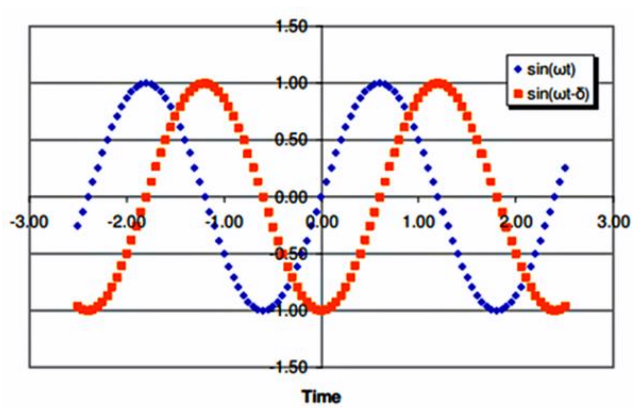
Consider $x = A\sin(\omega t - \delta)$. For $\delta = \dfrac{\pi}{2}$, we have

$$x = A\sin\left(\omega t - \frac{\pi}{2}\right)$$

$$x = A\left(\sin\omega t\cos\frac{\pi}{2} - \cos\omega t\sin\frac{\pi}{2}\right)$$

$$x = A\left(\sin\omega t(0) - \cos\omega t(1)\right)$$

$$x = -A\cos\omega t$$



Dynamics of Simple Harmonic Motion

Suppose an object, of mass m, is undergoing simple harmonic motion along the x-axis. This means that the object's position follows a single sinusoidal function, whose general form is $x(t) = A\sin(\omega t + \delta)$ or $x(t) = A\cos(\omega t + \delta)$.There is a kind of force that produces this kind of motion. Since we know $x(t)$, we can find $a(t)$. From Newton's second law of motion, the force is just $F(t) = ma(t)$.

The velocity is found by differentiating $x(t)$:

$$v(t) = \omega A\cos(\omega t + \delta)$$

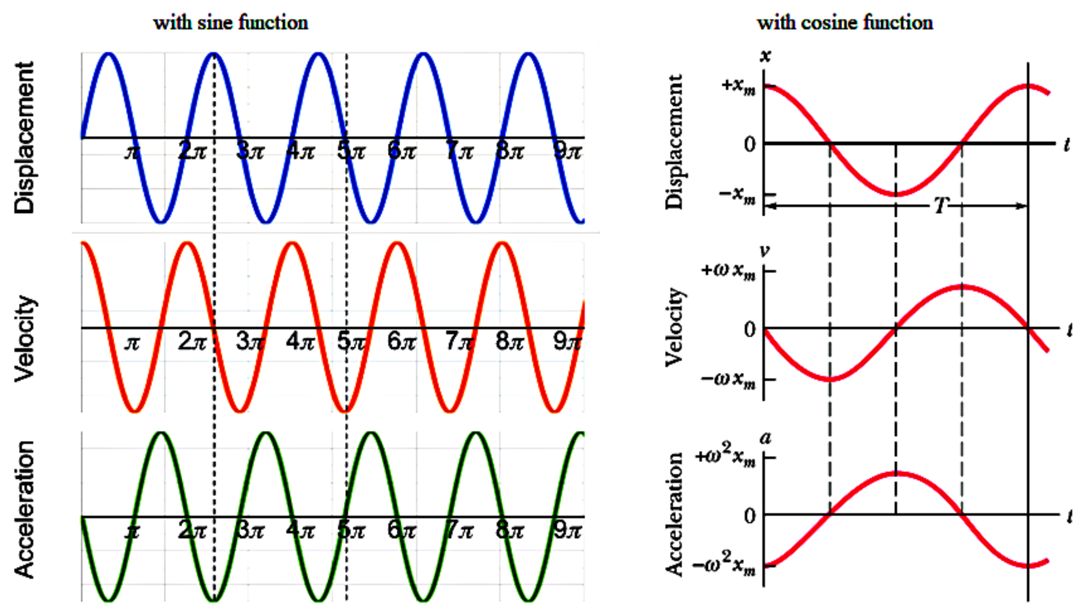
Differentiating again to get $a(t)$:

$$a(t) = -\omega^2 A\sin(\omega t + \delta) \qquad 7a$$

Since $x(t) = A\sin(\omega t + \delta)$, this simplifies to:

$$a(t) = -\omega^2 x(t) \qquad 7b$$

Equation (7) shows that the acceleration is proportional to the object's displacement from equilibrium position. The negative sign means that if $x > 0$ then $a < 0$, and if $x < 0$ then $a > 0$. This means that the acceleration is always towards the origin.



Substituting Equation (7b) into $F(t) = ma(t)$, we have

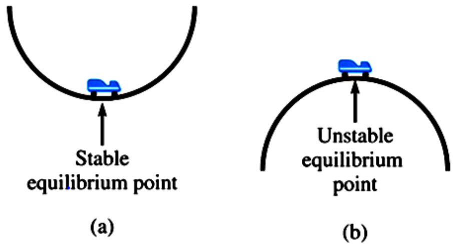
$$F = -m\omega^2 x \qquad 8$$

Equation (8) indicates that the force is always towards the origin, and is proportional to x. This type of force is called a ***"linear restoring"*** force:

(i) It is a restoring force because of the minus sign. The force tries to "restore" the object back to $x = 0$, the equilibrium position;

(ii) It is a linear force because its magnitude is proportional to x.

Thus, a linear restoring force produces a simple harmonic motion. Hence, simple harmonic motion occurs when the restoring force (the force directed toward a stable equilibrium point) is proportional to the displacement from equilibrium position. Equilibrium positions are points of ***stable equilibrium***, in which a small displacement from equilibrium points causes forces which tend to restore the system to its position of equilibrium. In contrast, a system in ***unstable equilibrium*** would, if disturbed, move away from the position of equilibrium.



Characteristics of SHM

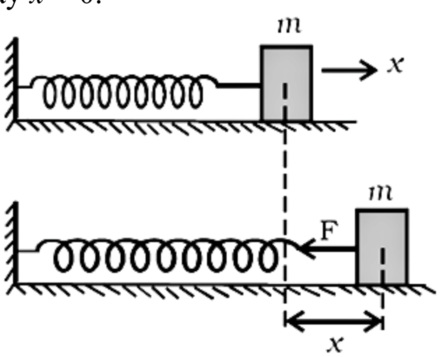
i. It is a repetitive (periodic) motion through a central equilibrium point.
ii. There is a symmetry of maximum displacement
iii. The period of each cycle is constant
iv. The linear restoring force causing the motion is directed toward the equilibrium point (minus sign).
v. The acceleration and restoring force are directly proportional to the displacement from the equilibrium point.

Simple Harmonic Oscillators

Any system that executes simple harmonic motion (SHM) is a simple harmonic oscillator (SHO).

The Simple Spring

Hooke's law works for virtually all stable systems, provided that the displacements from the equilibrium configuration are small. As long as Hooke's law is obeyed, then SHM will occur if the system is displaced from its equilibrium configuration and then released. Suppose an object of mass m is attached to one end of a spring, while the other end of the spring is held fixed. The object will have an equilibrium position, say $x = 0$.



If the object is displaced away from $x = 0$ the spring pulls (or pushes) it back to the equilibrium position ($x = 0$). Thus, the spring produces a restoring force. Hooke's law shows that the restoring force of a spring is approximately linear i.e. $F_s \approx -kx$. By comparing $F_s \approx -kx$ with $F = -m\omega^2 x$ we have

$$k = m\omega^2$$

$$\omega = \sqrt{\frac{k}{m}} \qquad 9$$

Since the period, $T = 2\pi/\omega$, the period for the object's motion for the spring force is:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad 10$$

In general, if the force on an object moving in one dimension, is of the form $F_x = -Cx$, then the motion will be simple harmonic with displacement $x(t) = A\sin(\omega t + \delta)$ where $\omega = \sqrt{C/m}$.

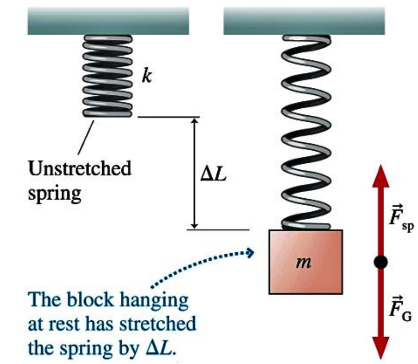
For a simple spring, the period T only depends on the mass and the restoring force constant. It does not depend on the amplitude A of the motion.

The final equation of motion for the object attached to a spring is:

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \delta\right) \qquad 11$$

Spring in Vertical Oscillations

(I) At equilibrium (no net force), spring is stretched (cf. horizontal spring): spring force balances gravity.

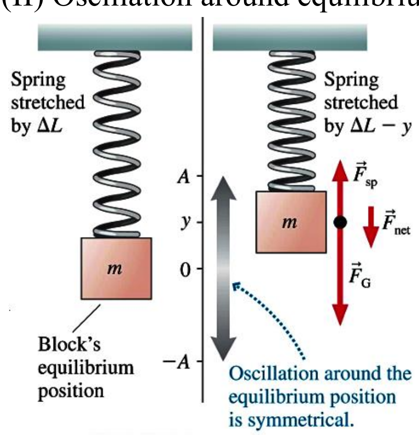


From Hooke's law: $F_S = -k\Delta y = +k\Delta L$

From Newton's law: $F_{net} = F_S + F_G = k\Delta L - mg = 0$

$$k\Delta L = mg \Rightarrow \Delta L = \frac{mg}{k}$$

(II) Oscillation around equilibrium, $y = 0$ (spring stretched), block moves upward, spring still stretched



$$F_{net} = F_S + F_G = k(\Delta L - y) - mg$$

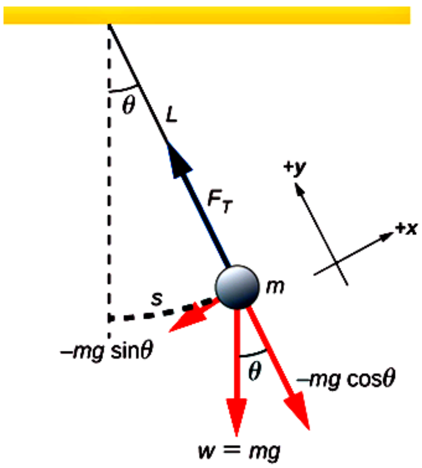
$$F_{net} = k\Delta L - ky - mg = k\Delta L - mg - ky$$

Using equilibrium equation: $k\Delta L - mg = 0$

$$F_{net} = -ky$$

The Simple Pendulum

A simple pendulum is constructed by attaching a mass to a thin rod or a light string. Ideally, it is a "*point*" mass attached to a massless string which is fixed to a fixed (rigid) point. If the pendulum is displaced from the equilibrium position, it swings back and forth, and its motion is periodic.



If the angle θ is measured in radians, $s = L\theta$.The component of gravity in the direction of the arc is $-mg\sin\theta$ or $-mg\sin(s/L)$. The negative sign means that the force of gravity is a restoring force.

$$F_s = -mg\sin\left(\frac{s}{L}\right) \tag{12}$$

For a force to produce simple harmonic motion, the force must be proportional to $-s$. Equation (12) is **not** sinusoidal because the force is proportional to $-\sin(s/L)$. Thus, the motion of a simple pendulum is **not** simple harmonic. A <u>sinusoidal</u> restoring force does not produce perfect sinusoidal motion. Only a <u>linear</u> restoring force gives perfect sinusoidal motion. If θ or s/L is small, less than 0.1 rad, then $\sin(s/L) \approx s/L$. Hence

$$F_s = -\frac{mg}{L}s \tag{13}$$

Thus, for small angles the restoring force is proportional to displacement, and the resulting motion is simple harmonic. Comparing with $F = -m\omega^2 x$,

$$m\omega^2 = \frac{mg}{L}$$

$$\omega = \sqrt{\frac{g}{L}} \tag{14}$$

Since $T = 2\pi/\omega$, we have

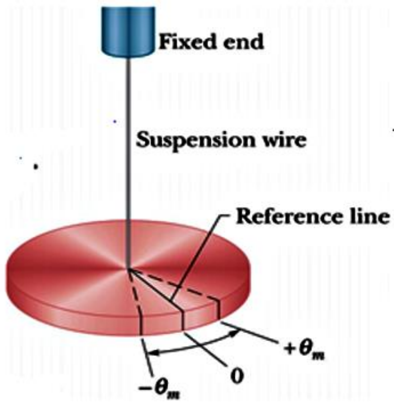
$$T = 2\pi\sqrt{\frac{L}{g}} \qquad 15$$

For small oscillations, the position x along the arc of the path is

$$x(t) = A\sin\left(\sqrt{\frac{g}{L}}t + \delta\right) \qquad 16$$

The Torsional Pendulum (Oscillator)

A horizontal disk suspended at its center by a thin fiber (rod or wire) forms a type of torsional oscillator. When the object is turned through an angle and released, it will oscillate back and forth. The analogies between linear and angular quantities make it easy to find an expression for the period of oscillations.



The twisted wire exerts a restoring torque τ on the object that is proportional to its angular position.

$$\tau = -K\theta \qquad 17$$

The magnitude of the torsion constant K depends on the material and dimensions of the wire.

There is an equivalent form of the general restoring force ($F = -m\omega^2 x$) of SHM in rotational motion, which is a restoring torque ($\tau = -I\omega^2\theta$).

$$\tau = -I\omega^2\theta \qquad 18$$

$$\therefore I\omega^2 = K$$

$$\omega^2 = \frac{K}{I} \Rightarrow \omega = \sqrt{\frac{K}{I}} \qquad 19$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{K}{I}} \qquad 20$$

Also, if the moment of inertia of the object about its axis of suspension, is I, then the angular acceleration α of the object due to the torque τ is

$$\alpha = \frac{\tau}{I} = -\frac{K}{I}\theta$$

Comparing with $a = -\omega^2 x$.

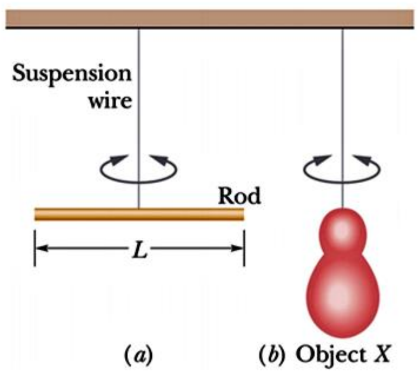
$$\omega^2 = \frac{K}{I} \Rightarrow \omega = \sqrt{\frac{K}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{K}}$$

To know the moment of inertia of an object attached to a torsion wire or rod, measure the torsion constant K and the period of oscillation T, and then use Equation (20) to find I.

Determination of the moment of inertia of an irregularly shaped object

We can determine the rotational inertia of an irregularly shaped object using a torsion pendulum. The period T of a system consisting of a suspension wire and any object depends on the rotational inertia I and the torsion constant K of the wire.



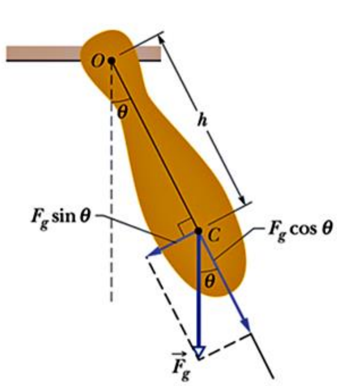
(i) Measure T_a (time for certain number of oscillations) for object with definite shape and hence known rotational inertia I, e.g., bar of length L rotated about its center $\left(I = \frac{1}{12}ML^2\right)$.

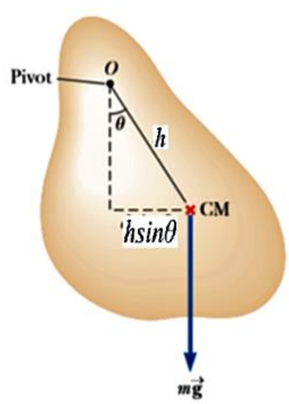
(ii) Measure T_b (time for the same number of oscillations as the definite shape) for the irregularly shaped object.

(iii) Since $T^2 \propto I \Rightarrow \frac{T^2}{I} = \text{constant}$. Use $I_b = \frac{T_b^2}{T_a^2} I_a$ to calculate the rotational inertia of the irregularly shaped object.

The Physical Pendulum

Any rigid body of any shape suspended at some point other than its center of gravity will oscillate when displaced from equilibrium position. Such an object is called a physical pendulum. The formula for the period of a simple pendulum also applies to a physical pendulum provided the length L is properly interpreted. Consider a physical pendulum pivoted about a horizontal axis at O. The pendulum is displaced so that the line from O to its center of gravity is at the angle θ from the vertical.





The pendulum's weight *mg* acts from the center of gravity and produces the restoring torque $\tau = mg \times h\sin\theta = mgh\sin\theta$. But for small angles of oscillation $\sin\theta \approx \theta$.

$$\tau = -mgh\theta \qquad 21$$

$$\tau = -I\omega^2\theta \qquad 22$$

Equation (22) is the rotational motion equivalent of $F = -m\omega^2 x$. *I* is the moment of inertia of the pendulum about *O*.

$$I\omega^2 = mgh \qquad 23$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}} \qquad 24$$

The simple pendulum whose period is the same as that of a given physical pendulum may be found by equating the formulas for the respective periods of oscillation

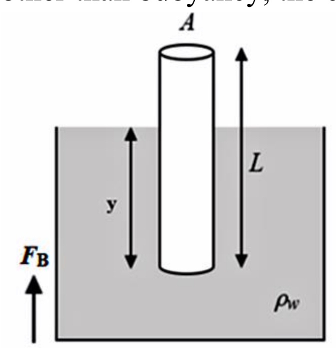
$$2\pi\sqrt{\frac{L'}{g}} = 2\pi\sqrt{\frac{I}{mgd}}$$

$$L' = \frac{I}{md} \qquad 25$$

L' is the ***equivalent length*** of a physical pendulum

Floating Object Oscillating in a Fluid

A cylinder of mass *m* and cross-sectional area *A* floats upright in a fluid of density ρ. Ignoring all forces other than buoyancy, the cylinder will execute SHM vertically when disturbed.



The restoring force is the buoyant force F_B

$$F_B = -m_{fd}g \qquad 26$$

Using $m_{fd} = \rho V_{fd}$ and $V_{fd} = Ay$, where ρ is the density of liquid, V_{fd} is the volume of fluid displaced, we have

$$F_B = -A\rho g y \qquad 27$$

Comparing Equation (27) with $F_x = -m\omega^2 x$ gives

$$m\omega^2 = A\rho g \Rightarrow \omega^2 = \frac{A\rho g}{m}$$

$$\omega = \sqrt{\frac{A\rho g}{m}} \qquad 28$$

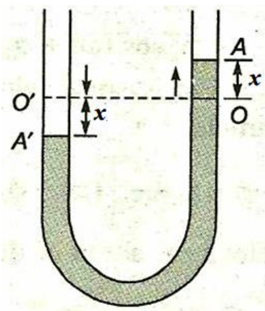
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{A\rho g}} \qquad 29$$

If the length of the cylinder is L and its density is ρ_o, $\rho_o = \frac{m}{V} = \frac{m}{AL}$. Therefore, $m = \rho_o AL$

$$T = 2\pi\sqrt{\frac{\rho_o AL}{A\rho g}} = 2\pi\sqrt{\frac{\rho_o L}{\rho g}} \qquad 30$$

Fluid Oscillating in a U-Tube

Consider a U-tube of uniform cross-sectional area A filled with a liquid of density ρ. The level of the liquid is vibrating up and down along its mean position OO' as shown in the figure.



The liquid in one arm is depressed by an amount x. The additional thrust of the liquid of length $2x$ in one of the arms tends to move the liquid level.

The restoring force = weight of the liquid column of length $2x$

$$F = mg = -A(2x)\rho g \qquad 31$$

$$F = -(2A\rho g)x \qquad 32$$

The total mass of the liquid = $AL\rho$, where L is the total length of the liquid column.

The liquid in the U-tube oscillates such that its acceleration a [from $F = ma = -(2A\rho g)x$ and $m = AL\rho$] is given by the expression

$$a = -\left(\frac{2g}{L}\right)x \qquad 33$$

Comparing Equation (32) with $F = -m\omega^2 x$, we have

$$-m\omega^2 x = -2A\rho g x$$

$$\omega^2 = \frac{2A\rho g}{m} \qquad 34a$$

$$\omega = \sqrt{\frac{2A\rho g}{m}} \qquad 34b$$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2A\rho g}} \qquad 35$$

Substituting $m = \rho AL$ into Equation (35), gives

$$T = 2\pi\sqrt{\frac{\rho AL}{2A\rho g}} = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{L/2}{g}} = 2\pi\sqrt{\frac{h}{g}} \qquad 36a$$

where h is the height of the of liquid column in one arm of the U-tube when the liquid is at rest or in equilibrium position.

$$T = 2\pi\sqrt{\frac{L}{2g}} = \pi\sqrt{\frac{4L}{2g}} = \pi\sqrt{\frac{2L}{g}} \qquad 36b$$

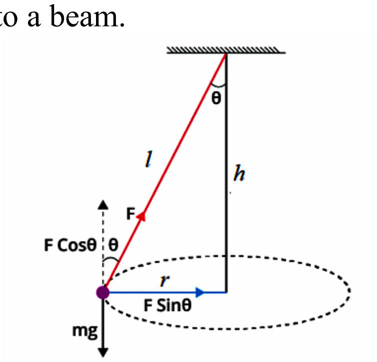
While the force constant ($C = 2\rho g A$) depends on acceleration g and the density ρ of the fluid because of the gravitational origin of the restoring force. T is planet-dependent again for the same reason, but is independent of fluid density and tube cross-section, as well as oscillation amplitude.

Summary of the basic features of simple harmonic oscillators (SHOs)

S/N	SHO	Restoring Force ($F = -Cx$)	Constant of Motion (C)	Angular Frequency (ω)	Period (T)
1	Simple spring	Spring force $F = -kx$	k	$\sqrt{\frac{k}{m}}$	$2\pi\sqrt{\frac{m}{k}}$
2	Simple pendulum	Gravitational force $F = -\frac{mg}{L}x$	$\frac{mg}{L}$	$\sqrt{\frac{g}{L}}$	$2\pi\sqrt{\frac{L}{g}}$
3	Torsion pendulum	Torsion force $\tau = -K\theta$	K	$\sqrt{\frac{K}{I}}$	$2\pi\sqrt{\frac{I}{K}}$
4	Physical pendulum	Gravitational force $\tau = -mgh\theta$	mgh	$\sqrt{\frac{mgh}{I}}$	$2\pi\sqrt{\frac{I}{mgh}}$
5	Floating object in liquid	Buoyant force $F = -A\rho g x$	$A\rho g$	$\sqrt{\frac{A\rho g}{m}}$	$2\pi\sqrt{\frac{m}{A\rho g}}$
6	Liquid Oscillating in a U-Tube	Buoyant force or gravitational force $F = -2A\rho g x$	$2A\rho g$	$\sqrt{\frac{2A\rho g}{m}}$	$2\pi\sqrt{\frac{m}{2A\rho g}} = 2\pi\sqrt{\frac{L}{2g}}$

The Conical Pendulum

The arrangement in the diagram below is known as a conical pendulum (also called a circular pendulum). This pendulum performs a periodic motion and, not SHM. It is a type of pendulum that spins in a complete circle, instead of moving back and forth. It is made of a mass attached to a spring (or rod) that is attached to a beam.



The two forces acting on the object are: (i) the Earth's attraction or gravitational force $\boldsymbol{mg}$, acting vertically downward, and (ii) the tension (force $\boldsymbol{F}$), which acts along the string. The acceleration is horizontal, toward the center of the circle, and of magnitude v^2/r. Thus, the vertical component of the tension must balance the weight $\boldsymbol{mg}$. The horizontal component of the tension is the resultant centripetal force. Applying Newton's second law of motion, $\sum \mathbf{F} = m\mathbf{a}$ to the vertical and horizontal components gives

$$T\cos\theta - mg = 0$$

$$T\sin\theta = ma = \frac{mv^2}{r} \text{ or } T\sin\theta = mr\omega^2$$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/r}{mg} \Rightarrow \tan\theta = \frac{v^2}{rg}$$

$$v = \sqrt{rg\tan\theta} \tag{37}$$

But $v = r\omega$ and $\tan\theta = \dfrac{r}{h}$

$$v^2 = rg\tan\theta \Rightarrow r^2\omega^2 = rg\tan\theta$$

$$\omega^2 = \frac{rg\tan\theta}{r^2} = \frac{rg\times r/h}{r^2} = \frac{g\times r^2/h}{r^2} = \frac{g}{h}$$

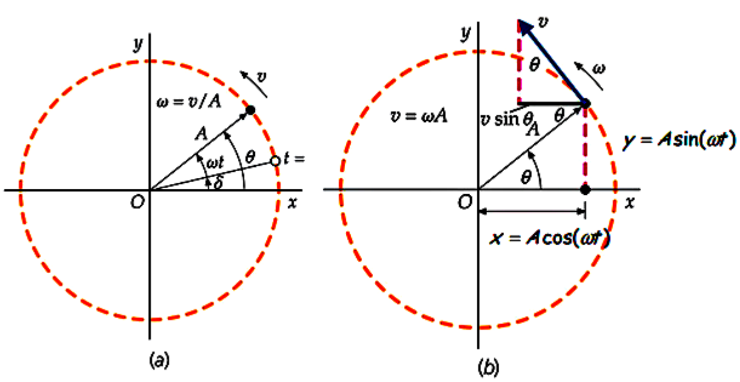
$$\omega = \sqrt{\frac{g}{h}}$$

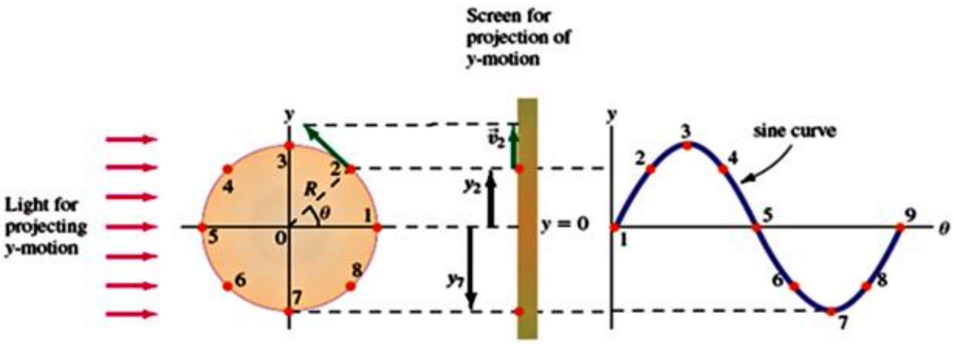
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

Simple Harmonic Motion and Uniform Circular Motion

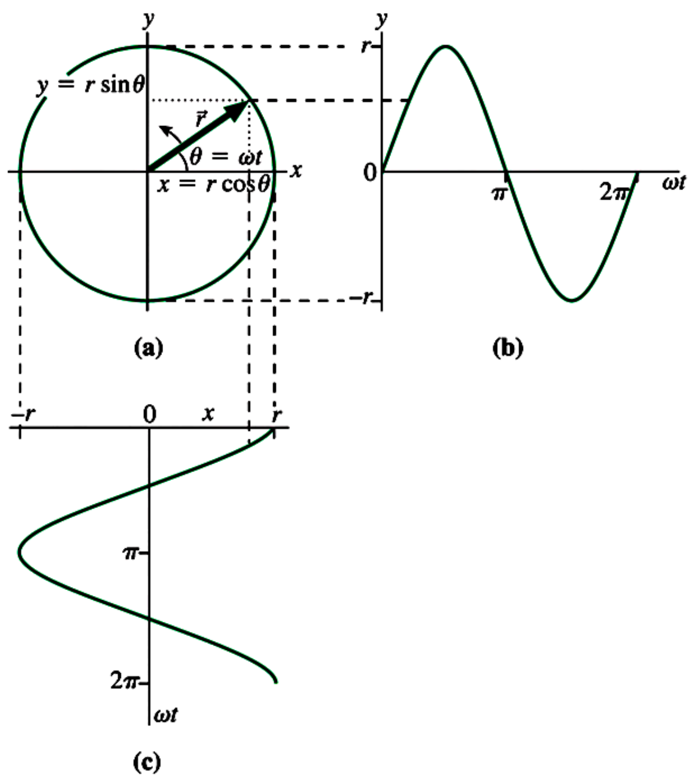
Circular motion is the superposition of two linear SHO that are 90° out of phase with each other. The Greek letter ω that is used to represent the angular frequency of an oscillator is also widely used to represent the angular speed of an object moving in a circle. This is no mere coincidence, rather it points to a deep association between one-dimensional SHM and uniform circular motion.

Consider an object in uniform circular motion about the origin of a cartesian coordinate system. The object is moving with constant angular speed ω at a fixed distance A from O.

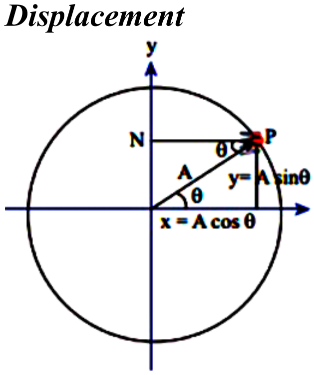




$$x = R\cos\theta = R\cos(\omega t + \delta)$$
$$y = R\sin\theta = R\sin(\omega t + \delta)$$



Displacement

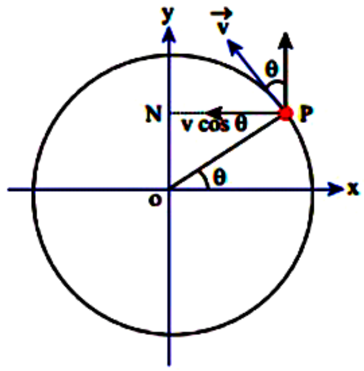


$x = A\cos\theta$

From uniform circular motion, $\theta = \omega t + \dfrac{1}{2}\alpha t^2$, with $\alpha = 0, \theta = \omega t$

$\therefore x = A\cos\omega t$

Velocity

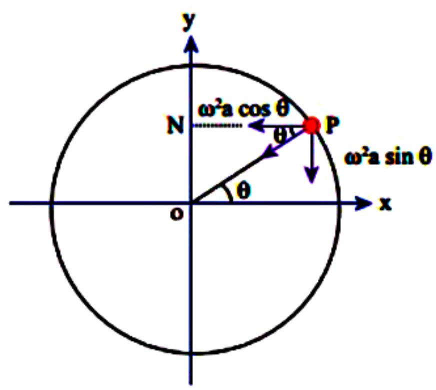


$v_x = -v\sin\theta = -v\sin\omega t$

But $v = r\omega = A\omega$

$\therefore v_x = -\omega A\sin\omega t$

Acceleration



$a_x = -a_c\cos\theta = -a_c\cos\omega t$

But $a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 = \omega^2 A$

$\therefore a_x = -\omega^2 A\cos\omega t$

Energy in Simple Harmonic Motion

For the simple spring system, the work done by the spring force when an object moves from position x_1 to position x_2 is given as:

$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} (-kx) dx$

$W = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$ 38

Using the work-energy theorem; $W_{Total} = \Delta K.E$. If the only force acting on the object is the force on the spring, then the total work is the work done by the spring. In this case we have:

$\frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

Rearranging the terms gives:

$\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2$ 39

The expression $\frac{1}{2}kx^2 + \frac{1}{2}mv^2$ is a constant of the motion. x is the position of the object and v is the speed of the object at the position x. As the object moves back and forth, the position x and the speed v both change but the combination $\frac{1}{2}kx^2 + \frac{1}{2}mv^2$ remains constant. The sum (potential energy and kinetic energy) is the total mechanical energy and always remains constant.

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \qquad 40$$

Substituting $x(t) = A\sin(\omega t + \delta)$ and $v(t) = \omega A\cos(\omega t + \delta)$ in equation (40), we have

$$E_T = \frac{1}{2}kA^2 \sin^2(\omega t + \delta) + \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \delta)$$

Since $k = m\omega^2$, we have

$$E_T = \frac{1}{2}kA^2 \left(\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right)$$

Since $\sin^2\theta + \cos^2\theta = 1$,

$$E_T = \frac{1}{2}kA^2 \qquad 41$$

In general, if the proportionality or force constant is C $(F_x = -Cx)$, the total mechanical energy is

$$E_T = \frac{1}{2}CA^2 \qquad 42$$

For example, in the case of a simple pendulum, $C = mg/l$, hence, $E_T = \frac{1}{2}\frac{mg}{l}x_{\max}$.

It can also be shown that

$$E_T = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta) + \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \delta)$$

But $v_{\max} = \omega A$

$$E_T = \frac{1}{2}mv_{\max}^2 \sin^2(\omega t + \delta) + \frac{1}{2}mv_{\max}^2 \cos^2(\omega t + \delta)$$

$$E_T = \frac{1}{2}mv_{\max}^2 \qquad 43$$

Hence, Equation (40) can be written as

$$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \qquad 44a$$

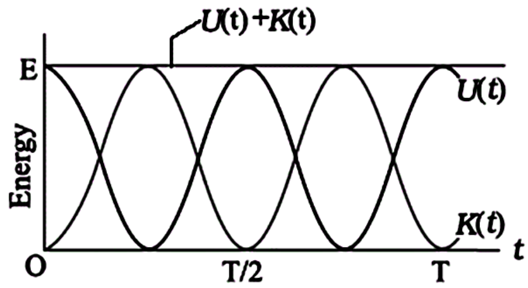
Or

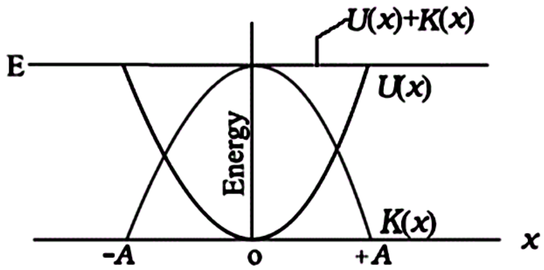
$$\frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \qquad 44b$$

(i) When the potential energy has its maximum values $\frac{1}{2}kA^2$, the kinetic energy is zero, and the object is instantaneously at rest at its position of maximum displacement, $x = A$.

(ii) When the kinetic energy has its maximum value $\frac{1}{2}mv_{\max}^2$, the potential energy is zero, and the object is moving with its maximum velocity at zero displacement.

(iii) At all other positions neither the potential energy nor kinetic energy is zero.



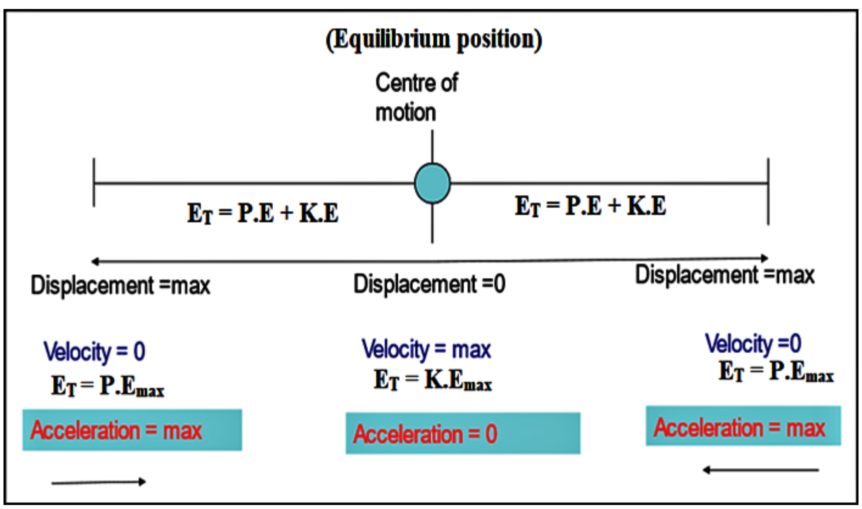


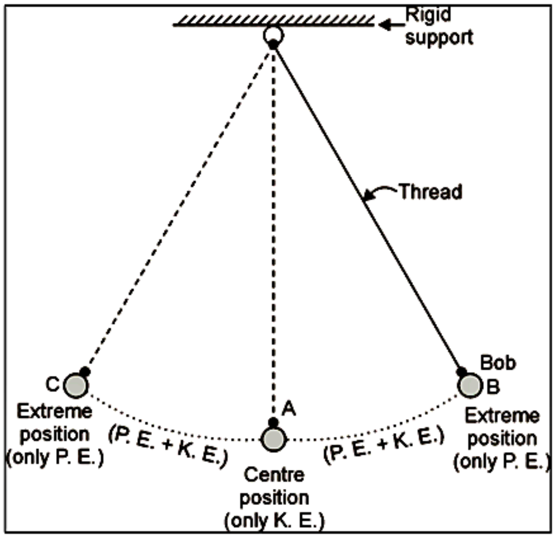
From Equation (44a), we get

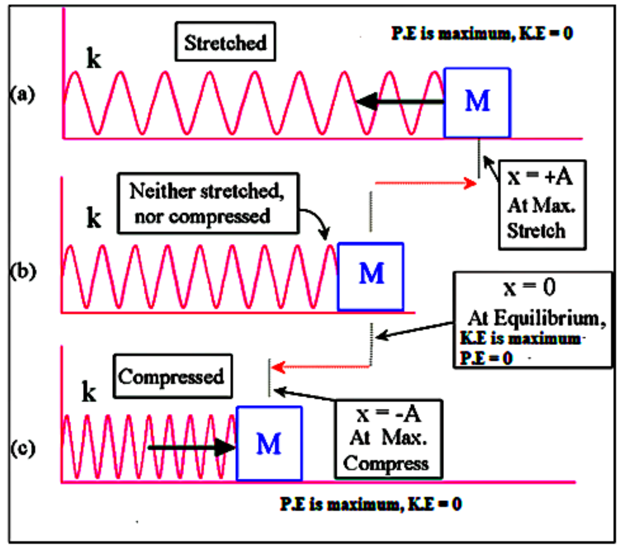
$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \omega\sqrt{\left(A^2 - x^2\right)} \qquad \text{45a}$$

From Equation (44b), we get

$$x = \sqrt{\frac{m}{k}\left(v_{\max}^2 - v^2\right)} = \frac{1}{\omega}\sqrt{\left(v_{\max}^2 - v^2\right)} \qquad \text{45b}$$

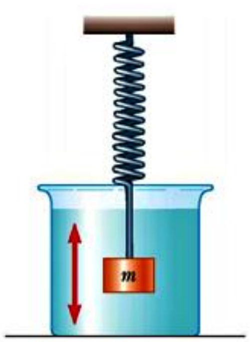


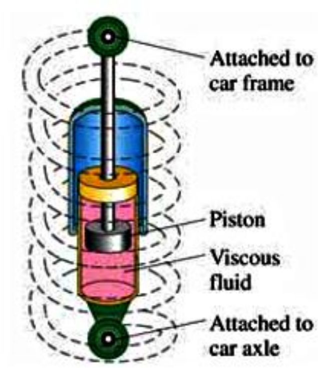


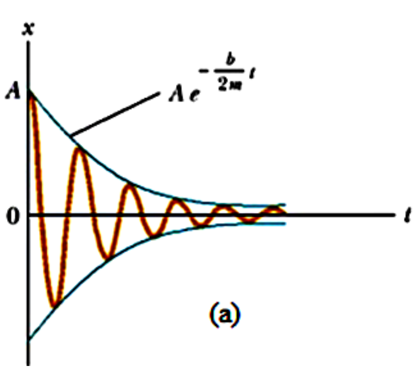


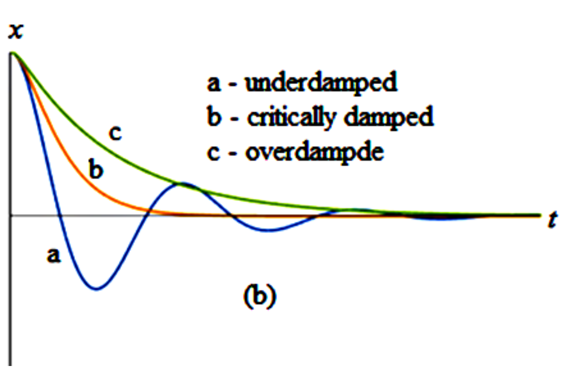
Damping of Simple Harmonic Motion

In real systems simple harmonic motion is subject to energy loss. Hence, all simple harmonic oscillations are damped to some degree due to energy dissipation via friction, viscous forces (drag force) or electrical resistance etc. The motion of damped systems is not conservative since energy is dissipated as heat. This energy loss process is called *damping*, and it is characterized by decreasing amplitude. The degree of damping may be classed as under-damped, critical-damped, or over-damped. The critically damped case corresponds to the most rapid return to equilibrium for a damped system. Critical damping is the desired situation for such systems as car suspensions, motor mounts on equipment, and meter movement in electrical instruments.









- Overdamped: The system returns to equilibrium without oscillating. Larger values of the damping the return to equilibrium slower.

- Critically damped: The system returns to equilibrium as quickly as possible without oscillating. This is often desired for the damping of systems such as doors.

- Underdamped: The system oscillates (with a slightly different frequency than the undamped case) with the amplitude gradually decreasing to zero.

The drag force exerted by air and liquids at low speed can be expressed as

$$\mathbf{F} = -b\mathbf{v} \qquad 46$$

where b is the damping constant or coefficient.
In the presence of a drag,

$$F = -bv - kx \Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad 47a$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega_0 x = 0 \qquad 47b$$

The solution to Equation (47b) is

$$x(t) = A_o e^{-\frac{b}{2m}t}\sin(\omega' t + \varphi) \qquad 48$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad 49$$

$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$	$\omega' = \sqrt{\omega_0^2\left[1 - \left(\frac{b}{2m\omega_0}\right)^2\right]} = \omega_0\sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2}$
Underdamping $\frac{b}{2m} < \omega_0$	$b_c = 2m\omega_0$
Critical damping $\frac{b}{2m} = \omega_0$	Underdamping $b < b_c$
Overdamping $\frac{b}{2m} > \omega_0$	Critical damping $b = b_c$
	Overdamping $b > b_c$

There are 4 different behaviors that depend on the damping constant b:
(i) No damping, $b = 0$: The motion reduces to SHM.
(ii) Underdamping, $0 < b < 2m\omega_0$: Decaying oscillations. A larger value of b leads to faster decay of oscillations. In the underdamped regime, the energy decays exponentially in time.

(iii) Overdamping, $2m\omega_0 < b$: Very slow monotonic decay.
(iv) Critical damping, $2m\omega_0 = b$: Quickest monotonic decay. Critical damping is the principle underlying shock absorbers.

Damping Parameter and Time Constant

The ratio $\gamma = \dfrac{b}{m}$ is the damping parameter. The amplitude of the damped oscillation at any given time is given by

$$A(t) = A_o e^{-\frac{1}{2}\gamma t} = A_o e^{-\frac{t}{2\tau}} \tag{50}$$

$$E(t) = \tfrac{1}{2}kA^2 = \tfrac{1}{2}k\left(A_o e^{-\frac{1}{2}\gamma t}\right)^2 = E_o e^{-\gamma t} = E_o e^{-\frac{t}{\tau}} \tag{51}$$

The ratio $\frac{m}{b}$ has units of time and is called the time constant $\tau = \dfrac{1}{\gamma}$. τ is the time for the energy to decrease by a factor of $\frac{1}{e}$. The time for amplitude to decrease by a factor of $\frac{1}{e}$ is $\dfrac{2}{\gamma} = 2\tau$. That is, it will take a time of 2τ for the amplitude to decrease by a factor of $\frac{1}{e}$.

Quality factor and Fractional Energy Loss

The damping of an oscillator is usually described by a dimensionless quantity Q called the quality factor or Q-factor. Q is simply a number which is a measure of damping. It is a convenient measure of how much longer the decay time is compared to the period. Lightly damped oscillations are referred to as high Q, and heaier damped oscillations – as low Q. Informally, the Q-factor represents the number of cycles completed by the oscillator before it "rings down" or "runs out" of energy.

$$Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per cycle/radian}}$$

The Q-factor of an oscillator relates to the fractional energy loss per cycle. The infinitesimal change in the energy is

$$dE = -\frac{1}{\tau}E_0 e^{-\frac{t}{\tau}}dt \tag{52}$$

If the energy loss per cycle, is small, we can replace dE with ΔE and dt with T (also $\omega' \approx \omega_0$).

$$-\frac{\Delta E}{E_0} = \frac{T}{\tau} = \frac{2\pi}{\omega_0 \tau} = \frac{2\pi}{Q} \tag{53}$$

Equation (53) gives the fractional energy loss per cycle, with the quality factor given as

$$Q = \omega_0 \tau = \frac{\omega_0}{\gamma} = \frac{\omega_0 m}{b} = \frac{2\pi}{(\Delta E/E)} \tag{54a}$$

In terms of amplitude decay, equation (54a) can be written as

$$Q = \frac{\pi}{(\Delta A/A)} \tag{54b}$$

Small damping means large Q, so an oscillator with small damping has a high quality.

Forced Oscillations and Resonance

Damping removes energy from a system executing SHM. It is possible to compensate for energy loss in damped systems by applying an external (driving) force that does positive work on the system. At any instant, energy can be put into the system by an applied force, that acts in the direction of motion of the oscillator.

- When an oscillating body is subjected to an external periodic force, it begins to oscillate under the action of the applied force.
- Initially, the body tries to oscillate with its natural frequency (ω_0), while the external force tries to impose its own frequency (ω_d) on the body.
- Thus, there is a kind of tussle between the driver (external force) and the driven (body) during which the amplitude of oscillation fluctuates (rises and falls) irregularly a number of times. This is "transient effect" which soon dies away.
- Eventually, the body yields to the external force and oscillates with constant amplitude and phase, and with the frequency of the applied force.
- Its oscillations are then called "forced oscillations". The oscillating body is called "driven harmonic oscillator" and the external force is called "driving force".
- The energy lost to damping in every cycle must be exactly compensated by work input from the external "driving force".

The simplest driving force may be described by $F_{ext} = F_o \sin\omega_d t$. F0 is its maximum value and ω_d is its angular frequency. This externa periodic force is added to the two forces (i.e. the restoring force, $F = -kx$ and the frictional force, $F = -bv = -b\dfrac{dx}{dt}$) considered in the damped oscillator case.

$$F(t) = -kx - bv + F_o \sin\omega_d t \qquad 55a$$

$$F + kx + bv = F_o \sin\omega_d t \qquad 55b$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin\omega_d t \qquad 55c$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin\omega_d t \Rightarrow \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin\omega_d t \qquad 55d$$

$\omega_0^2 = \dfrac{k}{m}$ is the angular frequency of the undamped oscillator ($b = 0$) and $\gamma = \dfrac{b}{m}$.

When the system oscillates at the frequency of the driving force, $x(t)$ must have a standard form for SHM i.e. $x(t) = A_0 \sin(\omega_d t \pm \delta_0)$ or $x(t) = A_0 \cos(\omega_d t \pm \delta_0)$.

Taking $x(t) = A_0 \sin(\omega_d t - \delta_0)$

$\dfrac{dx}{dt} = \omega_d A_0 \cos(\omega_d t - \delta_0)$ and $\dfrac{d^2x}{dt^2} = -\omega_d^2 A_0 \sin(\omega_d t - \delta_0)$

Substituting $x(t)$, $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$ in Equation (55d), we have

$$-\omega_d^2 A_0 \sin(\omega_d t-\delta_0)+\gamma A_0\omega_d \cos(\omega_d t-\delta_0)+\omega_0^2 A_0 \sin(\omega_d t-\delta_0)=\frac{F_0}{m}\sin\omega_d t \qquad 56$$

We can rewrite $\frac{F_0}{m}\sin\omega_d t$ as $\frac{F_0}{m}\sin\left(\omega_d t-\delta_0+\delta_0\right)=\frac{F_0}{m}\sin(\omega_d t-\delta_0)\cos\delta_0+\frac{F_0}{m}\cos(\omega_d t-\delta_0)\sin\delta_0$

$$(\omega_0^2-\omega_d^2)A_0 \sin(\omega_d t-\delta_0)+\gamma A_0\omega_d \cos(\omega_d t-\delta_0)=\frac{F_0}{m}\sin(\omega_d t-\delta_0)\cos\delta_0+\frac{F_0}{m}\cos(\omega_d t-\delta_0)\sin\delta_0 \quad *$$

If Equation (*) is to be satisfied for all values of t, then the coefficients of $\sin(\omega_d t-\delta_0)$ and $\cos(\omega_d t-\delta_0)$ on the R.H.S and L.H.S must be equal. Equating them we obtain

$$(\omega_0^2-\omega_d^2)A_0=\frac{F_0}{m}\cos\delta_0 \quad **$$

$$\gamma A_0\omega_d=\frac{F_0}{m}\sin\delta_0 \quad ***$$

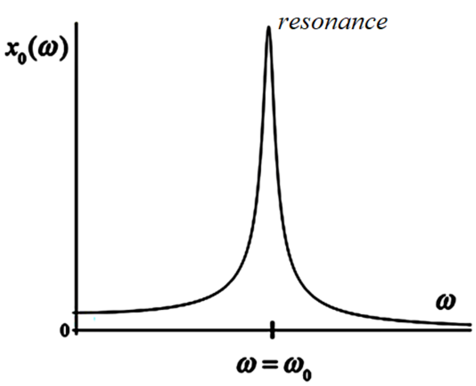
Squaring and adding Equations (**) and (***), we get

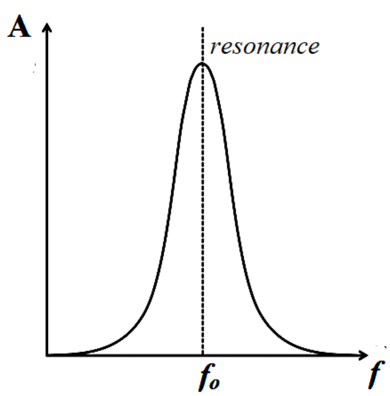
$$A_0=\frac{F_o/m}{\sqrt{\left(\omega_0^2-\omega_d^2\right)^2+\gamma^2\omega_d^2}}=\frac{F_o/m}{\sqrt{\left(\omega_0^2-\omega_d^2\right)^2+\left(\frac{b\omega_d}{m}\right)^2}} \qquad 57$$

Dividing Equation (***) by Equation (**), we get

$$\tan\delta_0=\frac{\gamma\omega_d}{\omega_0^2-\omega_d^2}=\frac{b\omega_d/m}{\omega_0^2-\omega_d^2} \qquad 58$$

The system oscillates at the angular frequency ω_d of the driving force. For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency ω_0 of the oscillating system. This increase in amplitude near ω_0 is called *resonance*. Hence ω_0 is also called resonance frequency of the system.





The amplitude of oscillations is generally not very high if f_{ext} differs much from f_0. As f_{ext} gets closer and closer to f_0, the amplitude of the oscillation rises dramatically.

Resonance and Power Transfer

The dissipation in the system, represented by "*b*" keeps the amplitude from going to infinity. The power input by the driving force is $P_{in}=F_{ext}v$, where $v=\omega_d A_0\cos(\omega_d t-\delta_0)$.

$$P_{in} = (F_0 \sin \omega_d t)\omega_d A_0 \cos(\omega_d t - \delta_0) \qquad \text{59a}$$

$$P_{in} = \omega_d A_0 F_0 \sin \omega_d t \cos(\omega_d t - \delta_0) \qquad \text{59b}$$

Expanding Equation (59b), we have

$$P_{in} = \omega_d A_0 F_0 \cos \delta_0 \sin \omega_d t \cos \omega_d t + \omega_d A_0 F_0 \sin \delta_0 \sin^2 \omega_d t \qquad \text{60a}$$

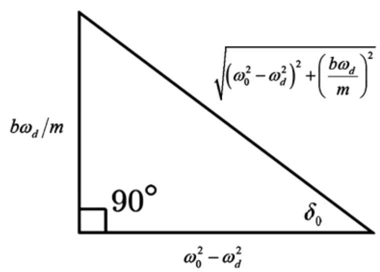
Since $\sin \omega_d t \cos \omega_d t = \sin 2\omega_d t$, we have

$$P_{in} = \omega_d A_0 F_0 \cos \delta_0 \sin 2\omega_d t + \omega_d A_0 F_0 \sin \delta_0 \sin^2 \omega_d t \qquad \text{60b}$$

The power input varies with time over a cycle. During one cycle, the first term in Equation (60b) is negative as often as it is positive and has an average value of zero. The average value of $\sin^2 \omega t$ over one cycle is $1/2$. Hence, the average power is therefore

$$P_{av} = \frac{1}{2}\omega_d A_0 F_0 \sin \delta_0 \qquad \text{61}$$

From $A_0 = \dfrac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + \left(\dfrac{b\omega_d}{m}\right)^2}} \Rightarrow \dfrac{A_0}{F_0} = \dfrac{1}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}}$



$$\sin \delta_0 = \frac{b\omega_d}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}} = \frac{1}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}} \times b\omega_d = \frac{A_0}{F_0} \times b\omega_d$$

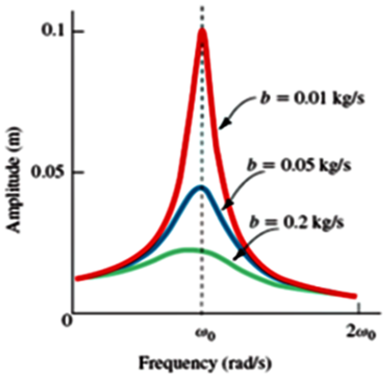
$$\sin \delta_0 = \frac{A_0}{F_0} \times b\omega_d \Rightarrow A_0 = \frac{F_0 \sin \delta_0}{b\omega_d} \qquad \text{62}$$

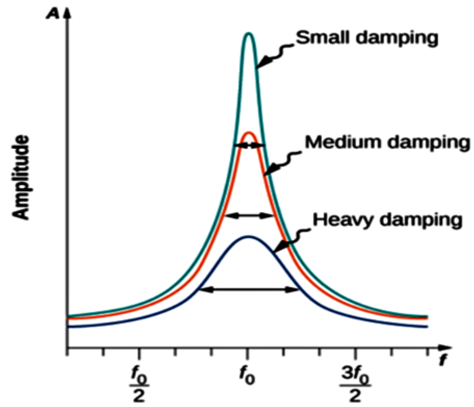
$$P_{av} = \frac{1}{2}\omega_d \left(\frac{F_0 \sin \delta_0}{b\omega_d}\right) F_0 \sin \delta_0 = \frac{F_0^2}{2b}\sin^2 \delta_0 \qquad \text{63}$$

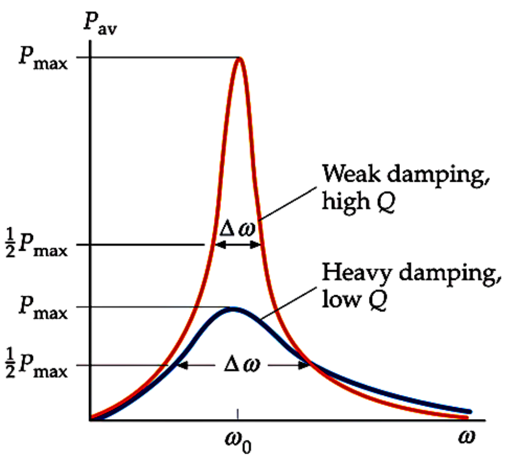
Equation (63) shows two important properties of a driven oscillator:

(i) The power input inversely proportional to the strength of the damping.

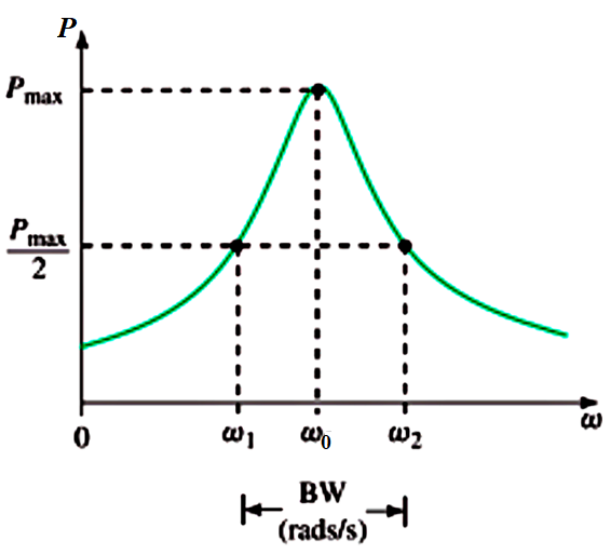
(ii) For variable driving frequency ω, the power input has its maximum value when $\delta_0 = \pm \pi/2$ or $\tan \delta_0 \to \pm\infty$. This occurs when $\omega = \omega_0$. This situation is called resonance. It is an important property of driven oscillations.

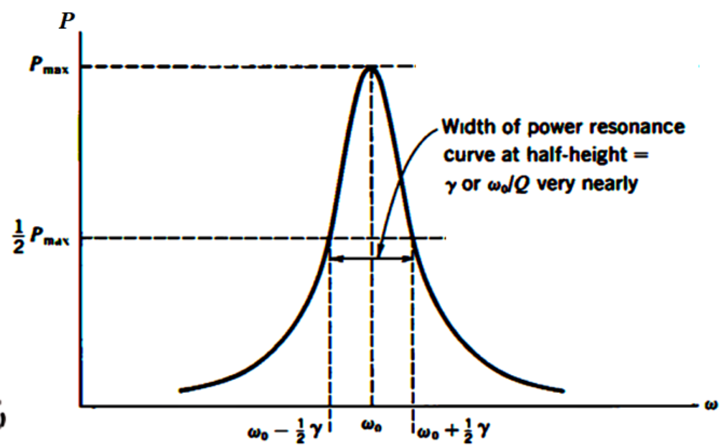






The above figure shows two curves of P_{av} vs ω, for the same values of F_0 and ω_0, but for two values of b differing by a factor of about 4. Small damping gives a sharp and high resonance peak, while large damping gives a broad and moderate resonance peak.





The power input is half its maximum value at frequencies ω_1 and ω_2 given as

$$\omega_1 = \omega_0 - \frac{\omega_0}{2Q} = \omega_0 - \frac{1}{2}\gamma \qquad 64$$

$$\omega_2 = \omega_0 + \frac{\omega_0}{2Q} = \omega_0 + \frac{1}{2}\gamma \qquad 65$$

where $BW = \gamma = \Delta\omega$

The full width at half maximum power is also given as

$$\omega_2 - \omega_1 = \frac{\omega_0}{Q} = \gamma \qquad 66a$$

$$\Delta\omega = \frac{\omega_0}{Q} = \gamma \qquad 66b$$

$$Q = \frac{\omega_0}{\Delta\omega} \qquad 66c$$

Q is a measure of how sharp the resonance is. The resonant amplitude is proportional to Q, whereas the width is inversely proportional to Q.

Worked Examples

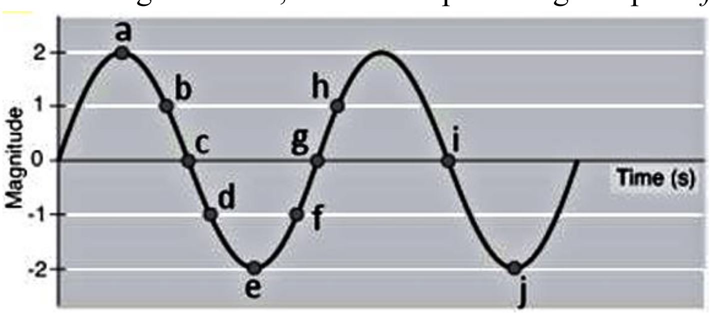
1. You measure your pulse to be 82 beats per minute. What is the period of your heartbeat in seconds?

A. 0.012 s B. 0.017 s C. 0.73 s D. 1.37 s

Solution:

$$T = \frac{1}{f} = \frac{1}{82\text{beats/min}} = \frac{60 \text{ s}}{82\text{beats}} = 0.73 \text{ s}$$

2. In the figure below, what is the phase angle at point *f* in degrees?



A. 315° B. 45° C. 225° D. 135°

Solution:

Point c is 180°, point e is 270°, and point g is 360°. Point f must therefore be 270° + 45° = 315°.

3. If ω = 640°/s and φ = 45°, at what time will the phase of the oscillator be 360°?

A. 0.125 s B. 0.492 s C. 1.78 s D. 2.03 s

Solution:

$$\theta = \omega t + \phi$$

$$t = \frac{\theta - \phi}{\omega} = \frac{360^{\circ} - 45^{\circ}}{640^{o}/\text{s}} = 0.492\text{s}$$

4. Show that $x(t) = C\cos\omega t + D\sin\omega t = A\cos(\omega t + \phi)$ where $A = \left(C^2 + D^2\right)^{\frac{1}{2}} > 0$, and $\phi = \tan^{-1}\left(-D/C\right)$

.Solution:

$$A\cos(\omega t + \phi) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t$$

Comparing coefficients, we see that $C = A\cos\omega t$ & $D = -A\sin\omega t$

$\therefore C^2 + D^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$

We choose the positive square root to ensure that $A > 0$

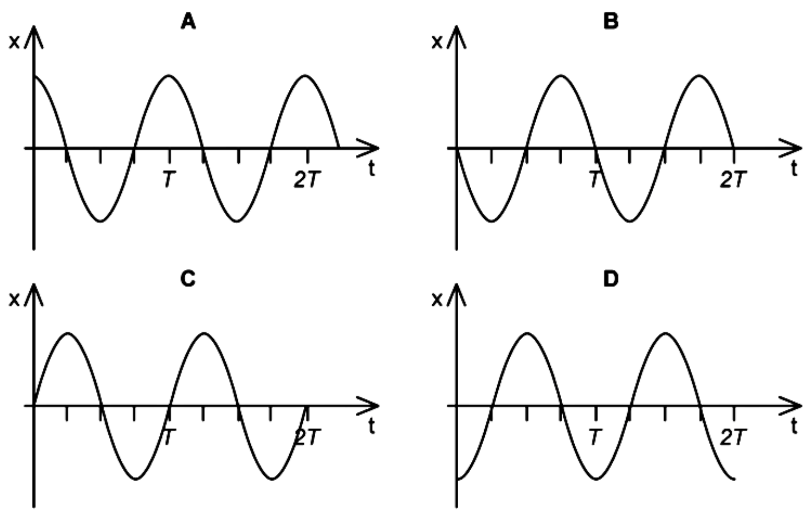
$A = \left(C^2 + D^2\right)^{\frac{1}{2}}$

$\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{-D/A}{C/A} = \dfrac{-D}{C}$

$\phi = \tan^{-1}\left(-D/C\right)$

Thus, $x(t) = C\cos\omega t + D\sin\omega t = A\cos(\omega t + \phi)$

5. An object oscillates with simple harmonic motion which can be described by the equation $x(t) = x_{\max}\cos\left(\omega t - \dfrac{\pi}{2}\right)$. Which of the following graphs correctly represents this equation?



Solution:

$x(t) = x_{\max}\cos\left(\omega t - \dfrac{\pi}{2}\right)$ is equivalent to $x(t) = x_{\max}\sin\omega t$

This describes an oscillation where $x = 0$ when $t = 0$. Hence, option A & D are not correct.
When ωt is positive, the oscillation will start moving in the +x direction. Hence, option B is not correct.
C is the correct option

6. A particle executes SHM of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant δ if the equation is written as $x = A\sin(\omega t + \delta)$

Solution:

We have $x = A\sin(\omega t + \delta)$. At $t = 0$. $x = A/2$, $\dfrac{A}{2} = A\sin\delta$

$\sin\delta = \dfrac{1}{2}$, $\delta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

The velocity is $v = \frac{dx}{dt} = \omega A\cos(\omega t + \delta)$

At t = 0, $v = \omega A\cos\delta$

Now, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

As v is positive at $t = 0$, δ must be equal to $\pi/6$.

7. A body makes angular SHM of amplitude $\pi/10$ rad and time period 0.05 s. If the body is at a displacement $\pi/10$ rad at $t = 0$, write the equation of the angular displacement as a function of time.

Solution:

Let the required equation be of the form $\theta = \theta_0 \sin(\omega t + \delta)$

Here, amplitude $\theta_0 = \frac{\pi}{10}$ rad

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.05\text{ s}} = 40\pi\text{ s}^{-1}$$

$$\therefore \theta = \frac{\pi}{10}\sin(40\pi t + \delta)$$

At $t = 0$, $\theta = \frac{\pi}{10}$ rad.

$$\frac{\pi}{10} = \frac{\pi}{10}\sin\delta \Rightarrow \sin\delta = 1$$

$$\delta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{10}\sin\left(40\pi t + \frac{\pi}{2}\right) = \frac{\pi}{10}\cos 40\pi t$$

8. A particle of mass 0.50 kg executes SHM under a force $F = -(50\text{ N/m})x$. If it crosses the center of oscillation with a speed of 10 m/s, find the amplitude of the oscillation.

Solution;

$v_{\max} = 10\text{ m/s}$, $k = 50\text{ N/m}$ & for a spring system $\omega = \sqrt{\frac{k}{m}}$

$$v_{\max} = \omega A \Rightarrow A = \frac{v_{\max}}{\omega}$$

$$A = \frac{v_{\max}}{\sqrt{\frac{k}{m}}} = \sqrt{\frac{m}{k}}v_{\max} = \sqrt{\frac{0.50\text{ kg}}{50\text{ N/m}}} \times 10\text{ m/s} = 1\text{ m}$$

Or

The kinetic energy of the particle when it is at the center of oscillation (x = 0) is the total energy $E_T = \frac{1}{2}mv_{\max}^2$ and the potential energy at maximum displacement A is the total energy $E_T = \frac{1}{2}kA^2$

$$E_T = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_{\max}^2 \Rightarrow A^2 = \frac{m}{k}v_{\max}^2$$

$$A = \sqrt{\frac{m}{k}}v_{\max}$$

9. A 1 kg mass attached to a spring with a spring constant of 36 N/m is stretched a distance of 0.01 m and released from rest. Calculate (i) the amplitude of the oscillation (ii) its maximum speed (iii) its maximum acceleration.

Solution:

(i) Amplitude A = 0.01 m

(ii) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{36}{1}} = 6\,\mathrm{s^{-1}}$

$v_{\max} = \omega A = 6\mathrm{s^{-1}} \times 0.01\mathrm{m} = 0.06\,\mathrm{m/s}$

(iii) $a_{\max} = \omega^2 A = (6\mathrm{s^{-1}})^2 \times 0.01\mathrm{m} = 0.36\ \mathrm{m/s^2}$

10. A pendulum set up in the stairwell of a 10-story building consists of a heavy weight suspended on a 34.0-m wire. What is the period of oscillation ($g = 9.81$ m/s^2)?

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{34m}{9.81m/s}} = 11.70s$$

11. A grindstone is suspended by a wire from its center. When a torque of 0.15 Nm is applied to the grindstone, it turns through 10°, and when it is released, it oscillates with a period of 1.0 s. Find the moment of inertia of the grindstone.

Solution:

$1^o = 0.01745$ rad, $\theta = 0.1745$ rad.

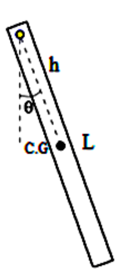
$$K = \frac{\tau}{\theta} = \frac{0.15\mathrm{Nm}}{0.1745\mathrm{rad}} = 0.86\,\mathrm{Nm/rad}$$

$$T = 2\pi\sqrt{\frac{I}{K}} \Rightarrow T^2 = \frac{4\pi^2 I}{K}$$

$$I = \frac{KT^2}{4\pi^2} = \frac{(0.86\mathrm{Nm/rad})(1.0\mathrm{s})^2}{4\pi^2} = 0.022\,\mathrm{kg/m^2}$$

12. An iron bar 80 cm long is suspended from one end. What is the period of its oscillation? What would be the length of a simple pendulum with the same size? Take g = 9.8 m/s^2.

Solution:



$I = \frac{1}{3}ML^2$

$h = \dfrac{L}{2}$

$$T = 2\pi\sqrt{\frac{I}{Mgh}} = 2\pi\sqrt{\frac{\frac{ML^2}{3}}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$T = 2\pi\sqrt{\frac{2(0.8\text{m})}{3(9.8\text{ms}^{-2})}} = 1.47\text{s}$$

$$L' = \frac{I}{Mh} = \frac{\frac{ML^2}{3}}{M\frac{L}{2}} = \frac{2L}{3} = \frac{2(0.8\text{m})}{3} = 0.533\text{m} = 53.3\text{ cm}$$

13. A uniform rod of length 1.00 m is suspended through one end and is set into oscillation with small amplitude under gravity. Find the period of oscillation.

Solution:

$$T = 2\pi\sqrt{\frac{I}{Mgh}} = 2\pi\sqrt{\frac{\frac{ML^2}{3}}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$T = 2\pi\sqrt{\frac{2\times 1.00\text{ m}}{3\times 9.80\text{ m/s}^2}} = 1.64\text{ s}$$

14. A uniform disc of radius 5.0 cm and mass 200 g is fixed at its center to a metal wire, with the other end fixed with a clamp. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

Solution:

The moment of inertia of the disc about the wire is

$$I = \frac{1}{2}MR^2 = \frac{(0.2\text{kg})(5.0\times 10^{-2}\text{m})^2}{2} = 2.5\times 10^{-4}\text{kgm}^2$$

The period is given by

$$T = 2\pi\sqrt{\frac{I}{K}} \Rightarrow K = \frac{4\pi^2 I}{T^2}$$

$$K = \frac{4\pi^2(2.5\times 10^{-4}\text{kgm}^2)}{(0.20\text{ s})^2} = 0.25\text{ kgm}^2\text{s}^{-2}$$

15. A 0.500-kg mass is vibrating in a system in which the restoring constant is 100 N/m; the amplitude of vibration is 0.200 m. Find: (i) the energy of the system. (ii) the maximum kinetic energy and maximum velocity. (iii) the potential energy and kinetic energy when $x = 0.100$. (iv) the maximum acceleration. (v) the equation of motion if $x = A$ at $t = 0$

Solution:

(i) Total energy of the system $E_T = \frac{1}{2}kA^2 = \frac{1}{2}(100)(0.200)^2 = 2.00\,\text{J}$

(ii) $K.E_{\max} = E_T = 2.00\text{ J}$

$$E_T = \tfrac{1}{2} m v_{\max}^2 \Rightarrow v_{\max}^2 = \frac{2E_T}{m}$$

$$v_{\max}^2 = \frac{2\times 2.00}{0.500} = 8$$

$$v_{\max} = 2.83\,\text{m/s}$$

(iii) $P.E = \tfrac{1}{2} kx^2 = \tfrac{1}{2}(100)(0.100)^2 = 0.500$ J

$$E_T = P.E + K.E \Rightarrow K.E = E_T - P.E = 2.00 - 0.500 = 1.50\,\text{J}$$

(iv) $a_{\max} = -\omega^2 A = \dfrac{k}{m} A = \dfrac{100}{0.500}(0.200) = 40.0\,\text{m/s}^2$

(v) If $x = A$ at $t = 0$, assuming $\phi = 0$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.500}} = 14.14\,\text{rad/s}$$

$$\therefore x = 0.200\cos 14.14t\ \text{m}$$

16. A 2-kg object stretches a spring 10 cm when is hangs vertically in equilibrium. The object is then attached to the spring resting on a smooth table and fixed at one end. The object is held a distance 5 cm from the equilibrium position and released. Find the amplitude *A*, angular frequency ω, period *T*, and frequency *f* for the resulting simple harmonic motion. What is the maximum speed of the object?

Solution:

The force constant of the spring is determined from the first measurement. For the vertical-equilibrium case, $ky_0 = mg$.

$$k = \frac{mg}{y_0} = \frac{(2kg)(9.81\text{m/s}^2)}{0.1\text{m}} = 196\text{N/m}$$

We are given that the initial position of the object is $x_o = 5\,\text{cm}$ = 0.05 m.

$$A = 0.05\,\text{m}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{196\text{N/m}}{2\text{kg}}} = 9.90\,\text{rad/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{9.90\text{rad/s}} = 0.63\text{s}$$

$$f = \frac{1}{T} = \frac{1}{0.63\text{s}} = 1.58\text{Hz}$$

$$v_{\max} = \omega A = (9.90\text{rad/s})(0.05\text{m}) = 0.5\text{m/s}$$

17. A pendulum has a length of 50 cm. Find its period when it is suspended in: (i) a stationary elevator. (ii) an elevator falling at a constant velocity of 5 m/s. (iii) an elevator falling at a constant acceleration of 2 m/s^2. (iv) an elevator rising at a constant velocity of 5 m/s. (v) an elevator rising at a constant acceleration of 2 m/s^2

Solution:

(i) $T = 2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{\dfrac{0.5\text{m}}{9.8\text{ms}^{-2}}} = 1.42\text{s}$

(ii) The motion of the pendulum bob is unaffected by motion of its support at constant velocity, hence T = 1.42 s.

(iii) The downward force on the bob here is $m(g-a)$.

$$T = 2\pi\sqrt{\frac{l}{(g-a)}} = 2\pi\sqrt{\frac{0.5\text{m}}{(9.8-2)\text{ms}^{-2}}} = 1.59\text{s}$$

(iv) As in (b), T = 1.42 s.

(v) The downward force on the bob here is $m(g+a)$.

$$T = 2\pi\sqrt{\frac{l}{(g+a)}} = 2\pi\sqrt{\frac{0.5\text{m}}{(9.8+2)\text{ms}^{-2}}} = 1.29\text{s}$$

18. A pendulum has period of 0.9927 s in Tokyo, and 0.9924 s in Cambridge. What is the ratio of g in Tokyo and Cambridge?

Solution:

$$\omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{T_T}{T_C} = \frac{2\pi\sqrt{\dfrac{l}{g_T}}}{2\pi\sqrt{\dfrac{l}{g_C}}} = \sqrt{\frac{g_C}{g_T}}$$

$$\frac{g_C}{g_T} = \left(\frac{T_T}{T_C}\right)^2 = \left(\frac{0.9927}{0.9924}\right)^2 = 1.0006$$

Or $\dfrac{g_T}{g_C} = \dfrac{1}{1.0006} = 0.9994$

19. A tuning fork that sounds the tone musicians call Middle-C oscillates at frequency f = 262 Hz. If the amplitude of the fork's oscillation decreases by a factor of 3 in 4.0 s, what are (i) the time constant of the oscillation and (ii) the quality factor.

Solution:

(i) $A = A_o e^{-t/2\tau}$, $A = \dfrac{A_o}{3}$

$$\frac{A_o}{3} = A_o e^{-t/2\tau}$$

$$\ln\frac{1}{3} = -\frac{t}{2\tau} \Rightarrow -\ln 3 = -\frac{t}{2\tau}$$

$$\tau = \frac{t}{2\ln 3} = \frac{4.0\text{s}}{2\ln 3} = 1.82\text{s}$$

(ii) $Q = \omega\tau = 2\pi f\tau = 2\pi(262\text{Hz})(1.82\text{s}) = 2996$

20. A 2-kg object oscillates on a spring of force constant $k = 400$ N/m with initial amplitude of 3 cm.
(i) Find the period and the total initial energy. (ii) If the energy decreases by 1 % per period, find the damping constant b and Q-factor.

Solution:

(i) $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{2\text{kg}}{400\text{N/m}}} = 0.444\text{s}$

$E_o = \dfrac{1}{2}kA_o^2 = \dfrac{1}{2}(400\text{N/m})(0.03\text{m})^2 = 0.18\text{J}$

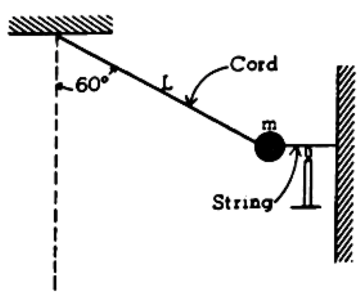
(ii) $\dfrac{\Delta E}{E} = 0.01$

$\dfrac{\Delta E}{E} = \dfrac{bT}{m} = 0.01$

$0.01 = \dfrac{b \times 0..444\text{s}}{2\text{kg}} \Rightarrow b = \dfrac{0.01 \times 2\text{kg}}{0.444\text{s}} = 0.045\text{kg/s}$

$\dfrac{\Delta E}{E} = \dfrac{2\pi}{Q} \Rightarrow Q = \dfrac{2\pi}{0.01} = 628$

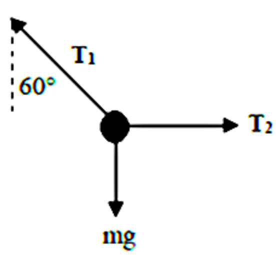
21. A pendulum consists of a small object of mass m fastened to the end of an inextensible cord of length L. Initially, the pendulum is drawn aside through an angle of 60° with the vertical and held by a horizontal string as shown in the diagram above. This string is burned so that the pendulum is released to swing to and fro.



(i) Draw a force diagram identifying all the forces acting on the object while it is held by the string. (ii) Determine the tension in the cord before the string is burned. (iii) Show that the cord, strong enough to support the object before the string is burned, is also strong enough to support the object as it passes through the bottom of its swing. (iv) The motion of the pendulum after the string is burned is periodic. Is it also simple harmonic? Why, or why not?

Solution:

(i)



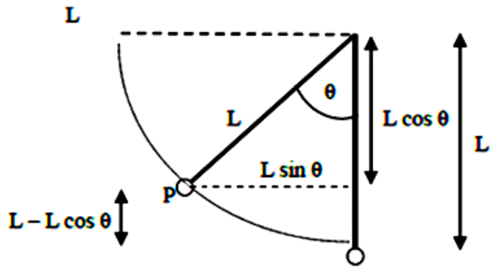
(ii) $\sum F_y = 0$

$T_1 \cos\theta - mg = 0$

$T_1 \cos\theta = mg$

$T_1 = \dfrac{mg}{\cos\theta} = \dfrac{mg}{\cos 60^o} = 2mg$

(iii) When the string is cut it swings from top to bottom with θ on the opposite side as shown below.



The potential energy at the top (maximum displacement) and the kinetic energy at the bottom (zero displacement) are equal (total energy).

$U_{top} = K_{bottom} \Rightarrow mgh = \dfrac{1}{2}mv^2$

$h = L - L\cos\theta = L(1-\cos\theta)$

$v = \sqrt{2gh} = \sqrt{2gL(1-\cos\theta)} = \sqrt{2gL(1-\cos 60^o)} = \sqrt{2gL(1-1/2)} = \sqrt{gL}$

The arc length formed during swinging is part of a circumference of a circle, hence there is a resultant force (centripetal force) acting on the object when it is at the equilibrium position.

As it passes through the bottom of its swing $\sum F_y = F_c = \dfrac{mv^2}{r} \Rightarrow T_1 - mg = \dfrac{mgL}{L}$

$T_1 - mg = mg \Rightarrow T_1 = 2mg$

Since it is the same force as before, it will be possible.

(iv) This motion is not simple harmonic because the restoring force $F = mg\sin\theta$, is not directly proportional to the displacement due to the sine function. However, for small angles of θ the motion is approximately SHM, though not exactly, but in this case the larger value of θ creates larger disparity.

22. A 1.5 kg mass is attached to the end of a 90 cm string. The system is whirled in a horizontal circular path. The maximum tension that the string can withstand is 400 N. What is the maximum number of revolutions per minute allowed if the string is not to break?

Solution:

$r = l$

$F_c = T = \dfrac{mv^2}{r} = \dfrac{mv^2}{l}$

$T = \dfrac{mv^2}{l} \Rightarrow v = \sqrt{\dfrac{Tl}{m}}$

$$v=\sqrt{\frac{400\text{N}\times 0.9\text{m}}{1.5\text{kg}}}=15.5\text{m/s}$$

$$v=r\omega=l\omega\Rightarrow\omega=\frac{v}{l}$$

$$T=\frac{2\pi}{\omega}=\frac{2\pi l}{v}=\frac{2\pi(0.9m)}{15.5\text{m/s}}=0.365\text{s}$$

$$T=\frac{t}{n}\Rightarrow n=\frac{t}{T}$$

For 1 minute, $n=\dfrac{60\text{s}}{0.365\text{s}}=164$

23. What is the time constant of an oscillator if its amplitude of oscillation is decreased to 30.0% of its original value in 21.5 s?

(A) 35.7 s (B) 8.93 s (C) 17.9 s (D) 3.16 s

Solution:

$$A=A_0e^{-\frac{1}{2}\gamma t}\Rightarrow 0.3A_0=A_0e^{-\frac{1}{2}\gamma(21.5)}$$

$$\ln 0.30=-\frac{1}{2}\gamma(21.5)$$

$$\gamma=\frac{2\ln 0.30}{-21.5}$$

$$\tau=\frac{1}{\gamma}=\frac{-21.5}{2\ln 0.30}=8.93\text{s}$$

Correct option is B

24. The displacement, *x*, of a particle from a fixed point, O, is given by $x=7\sin\omega t+24\cos\omega t$. ω being a constant.

(i) Show that the particle is describing simple harmonic motion about O; (ii) Calculate the amplitude of the motion.

Solution:

(i) Comparing $x=7\sin\omega t+24\cos\omega t$ with $x=\dfrac{v_0}{\omega}\sin\omega t+x_0\cos\omega t$, the motion is SHM.

(ii) $A=\sqrt{x_0^2+\dfrac{v_0^2}{\omega}}=\sqrt{24^2+7^2}=\sqrt{525}=25$

25. A damped oscillator loses one-twentieth of its energy during each cycle. (i) How many cycles elapse before half the original energy is dissipated? (ii) During this time by how much is the amplitude reduced? (iii) What is the *Q*-factor? (iv) If the natural frequency is *f* = 100 Hz, what is the width of the resonance when the oscillator is driven?

Solution:

(i) $E = E_0 e^{-\frac{t}{\tau}}$

Condition 1: $E = \frac{19}{20} E_0 @ t = T$

Condition 2: $E = \frac{1}{2} E_0$

$\ln \frac{19}{20} = -\frac{T}{\tau}$ ------------------ (i)

$\ln \frac{1}{2} = -\frac{t}{\tau}$ ----------------- (ii)

Dividing Equation (ii) by Equation (i)

$t = \left(\frac{\ln 0.5}{\ln 0.95} \right) T = 13.5T$

$t = nT \Rightarrow n = 13.5$

(ii) $A = A_0 e^{-\frac{t}{2\tau}} \Rightarrow \frac{A}{A_0} = e^{-\frac{1}{2} \times \frac{t}{\tau}}$

From Equation (ii), $\frac{t}{\tau} = -\ln 0.5$

$\frac{A}{A_0} = e^{-\frac{1}{2} \times -\ln 0.5} = 0.71$

Amplitude reduces by 0.29 or 29%

(iii) $\frac{\Delta E}{E_0} = \frac{2\pi}{Q} \Rightarrow \frac{1}{20} = \frac{2\pi}{Q}$

$Q = 40\pi = 126$

(iv) $Q = \frac{\omega_0}{\Delta\omega} \Rightarrow \Delta\omega = \frac{\omega_0}{Q}$

$\Delta\omega = \frac{2\pi f_0}{Q} = \frac{2\pi \times 100\text{Hz}}{40\pi} = 5$ Hz

26. A particle has displacement given by $x = 3\cos(5\pi t + \pi)$, where x is in meters and t in seconds. (i) What are the frequency f and period T of the motion? (ii) What is the greatest distance the particle travels from equilibrium? (iii) Where is the particle at time $t = 0$? At time $t = ½$ s.

Solution:

(i) $2\pi f = 5\pi \Rightarrow f = \frac{5}{2} = 2.5$ Hz

$T = \frac{1}{f} = \frac{1}{2.5} = 0.4$ s

(ii) $A = 3$ m

(iii) At $t = 0$ s, $x = 3\cos\left(5\pi(0) + \pi\right) = 3\cos\pi = -3$ m

At $t = ½$ s, $x = 3\cos\left(5\pi(\tfrac{1}{2}) + \pi\right) = 3\cos\dfrac{7\pi}{2} = 0$ m

27. A particle in SHM is at rest a distance of 6 cm from its equilibrium (rest) position at $t = 0$. Its period is 2 s. Write expressions for its position x, its velocity v, and its acceleration a as functions of time.
Solution:
$x@t = 0, x(0) = x_0 = 6\text{cm} \neq 0$, the appropriate function is cosine

$$x = A\cos\omega t = A\cos\frac{2\pi}{T}t = 6\cos\frac{2\pi}{2}t = 6\cos\pi t$$

$$v = \frac{dx}{dt} = -6\pi\sin\pi t$$

$$a = \frac{dv}{dt} = -6\pi^2\cos\pi t$$

28. A car's suspension acts like a mass-spring system with $m = 1200$ kg and $k = 5.8 \times 10^4$ N/m. Its shock absorbers provide a damping constant of $b = 230$ kg/s. After the car hits a bump in the road, how many oscillations will it make before the amplitude drops to half its initial value?
Solution:

$$A = A_0 e^{-\frac{b}{2m}t} \Rightarrow \frac{A_0}{2} = A_0^{-\frac{b}{2m}t}$$

$$0.5 = e^{-\frac{b}{2m}t} \Rightarrow \ln 0.5 = -\frac{b}{2m}t$$

$$t = -\frac{2m}{b}\ln 0.5 = -\frac{2\times1200}{230}\ln 0.5 = 7.23\text{s}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1200}{5.8\times10^4}} = 0.90\text{s}$$

$$n = \frac{t}{T} = \frac{7.23}{0.90} = 8$$

29. The equation of a particle executing SHM is given as $y = 5\sin\left(\pi t + \dfrac{\pi}{3}\right)$, y is in meters. Calculate (i) amplitude, (ii) period, (iii) maximum velocity, and (iv) velocity after 1 second.
Solution:
$y = 5\sin\left(\pi t + \dfrac{\pi}{3}\right)$ compare with $y = A\sin\left(\omega t + \delta\right)$

(i) Amplitude $A = 5$ m

(ii) Period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2\text{s}$

(iii) $v_{\max} = A\omega = 5\pi = 15.7\text{m/s}$

(iv) $v = \dfrac{dy}{dt} = 5\pi\cos\left(\pi t + \dfrac{\pi}{3}\right) = 5\pi\cos\left(\pi(1) + \dfrac{\pi}{3}\right) = 5\pi\cos\dfrac{4\pi}{3} = -7.85\text{m/s}$

30. A spring of angular frequency 8.08 s^{-1} is compressed 0.100 m from equilibrium $(x_0 = -0.100\text{m})$ but is given a shove to create a velocity in the +*x* direction of $v_0 = 0.400\text{m/s}$. Determine: (i) the amplitude *A*; (ii) the phase angle ϕ; (iii) the displacement *x*(*t*) as a function of time.

Solution:

(i) $A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{(-0.100)^2 + \frac{0.400^2}{8.08^2}} = 0.112\text{m}$

(iib) $\phi = \tan^{-1}\left(-\frac{v_0/\omega}{x_0}\right) = \tan^{-1}\left(-\frac{0.400/8.08}{-0.100}\right) = 26.34^o = 0.45\text{rad}$

(iii) $x(t) = A\cos(\omega t + \phi) = 0.112\cos(0.808t + 0.45)$

31. A particle in SHM is at a distance of 6 cm from its equilibrium position at *t* = 0. Its period is 2 s. Write expressions for its position *x*, its velocity *v*, and its acceleration *a* as functions of time.

Solution:

At $t = 0$, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\text{rad/s}$

$x = 6\cos\pi t$

$v = \frac{dx}{dt} = -6\pi\sin\pi t$

$a = \frac{dv}{dt} = -6\pi^2\cos\pi t$

32. The motion of an oscillating liquid in a U-tube is

A. periodic but not simple harmonic

B. periodic

C. simple harmonic and time period is independent of the density of the liquid

D. simple harmonic and time period is directly proportional to the density of the liquid

Solution: p

The motion of an oscillating liquid in a U-tube is SHM with the period, $T = 2\pi\sqrt{\frac{L}{2g}} = \sqrt{\frac{h}{g}}$,

where *h* is the height of the of liquid column in one arm of the U-tube in equilibrium position of the liquid. Therefore, *T* is independent of the density of the liquid.

Option C is correct.

33. An oscillator has a period of 3s. Its amplitude decreases by 5% each cycle. (i) what is the time constant? (ii) By how much does its energy decrease in each cycle (iii) What is the Q-facor?

Solution:

(i) $A = A_0 e^{-\frac{t}{2\tau}}$

$0.95A_0 = A_0 e^{-\frac{T}{2\tau}}$

$$0.95 = e^{-\frac{T}{2\tau}} \Rightarrow \tau = \frac{-T}{2\ln 0.95}$$

$$\tau = \frac{-3\text{ s}}{2\ln 0.95} = 29.24\text{ s}$$

(ii) $E = E_0 e^{-\frac{t}{\tau}} = E_0 e^{-\frac{T}{\tau}} = E_0 e^{-\frac{3}{29.24}} = 0.9025 E_0$

$\Delta E = 0.1 = 10\%$

(iii) $\dfrac{\Delta E}{E_0} = \dfrac{2\pi}{Q} \Rightarrow Q = \dfrac{2\pi}{\Delta E/E_0} = \dfrac{2\pi}{0.1} = 62.8$

34. A damped harmonic oscillator involves a block (m = 2 kg), a spring (k = 10 N/m), and a damping force $F = -bv$. Initially it oscillates with an amplitude of 0.25 m; because of the damping, the amplitude falls to three-fourths of its initial value after four complete cycles.

(i) What is the value of b? (ii). How much energy is lost during these four cycles?

Solution:

(i) $A = \frac{3}{4} A_0 \;\&\; t = 4T$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{10}} = 2.81\text{s}$$

$$A = A_0 e^{-\frac{b}{2m}t}$$

$$\frac{3}{4} A_0 = A_0 e^{-\frac{b}{2m}\times 4T}$$

$$\ln 0.75 = -\frac{4bT}{2m}$$

$$b = \frac{2m\ln 0.75}{-4T} = \frac{2\times 2\text{kg}\times \ln 0.75}{-4\times 2.81\text{s}} = 0.102\text{kg/s}$$

(ii) $E = E_0 e^{-\frac{b}{m}t} = E_0 e^{-\frac{0.102}{2}\times 4\times 2.81} = 0.56 E_0$

$$E_0 = \frac{1}{2}kA_0^2 = \frac{1}{2}\times 10\times (0.25)^2 = 0.3125\text{J}$$

$$\Delta E = E_0 - 0.56E_0 = 0.44E_0 = 0.44\times 0.3125 = 0.1375\text{ J}$$

35. For a child on a swing, the amplitude drops by a factor of 1/e in about eight periods if no additional mechanical energy is given to the system. Estimate the Q-factor for the system.

Solution:

$$A = A_o e^{-t/2\tau}$$

$$\frac{A}{A_o} = e^{-8T/2\tau}$$

$$\frac{A}{A_o} = \frac{1}{e} = e^{-1} = e^{-8T/2\tau}$$

$$1 = \frac{8T}{2\tau} \Rightarrow 2\tau = 8T$$

$$\tau = 4T$$

$$Q = \omega\tau = \frac{2\pi\tau}{T} = \frac{2\pi \times 4T}{T} = 8\pi$$

i.e. if the number of periods is *N*, then $Q = N\pi$.

36. A damped oscillator loses 2% of its energy each cycle. (i) How many cycles elapse before half its initial energy is dissipated? (ii) What is the *Q*-factor?

Solution:

$$E = 0.98E_0$$

$$\ln 0.98 = -\frac{T}{\tau} \text{ --------------- (i)}$$

$$E = 0.50E_0$$

$$\ln 0.50 = -\frac{t}{\tau} \text{ --------------- (ii)}$$

Dividing (ii) by (i), we have

$$\frac{\ln 0.50}{\ln 0.98} = \frac{t}{T} \Rightarrow t = \left(\frac{\ln 0.50}{\ln 0.98}\right)T$$

$$t = 34.3T = 34.3 \text{ cycles}$$

(ii) $\frac{\Delta E}{E} = \frac{2\pi}{Q} \Rightarrow \frac{2}{100} = \frac{2\pi}{Q}$

$$Q = 100\pi = 314$$

37. A 0.5 kg mass oscillations on a spring of force constant $k = 300$ N/m. During the first 10 s, it loses 0.5 J to friction. If the initial amplitude was 15 cm. (i) How long before the energy is reduced to 0.1 J? (ii) What is the frequency of oscillation?

Solution:

(i) $E_0 = \frac{1}{2}kA_0^2 = \frac{1}{2}(300 \text{ N/m})\left(\frac{15}{100}\right)^2 = 3.375 \text{ J}$

$$E = E_0 - 0.5 = 3.375 - 0.500 = 2.875 \text{ J}$$

$$E = E_0 e^{-\frac{t}{\tau}} \Rightarrow 2.875 = 3.375 e^{-\frac{10}{\tau}}$$

$$\frac{2.875}{3.375} = e^{-\frac{10}{\tau}} \Rightarrow \ln\left(\frac{2.875}{3.375}\right) = -\frac{10}{\tau}$$

$$\tau = -\frac{10}{\ln\left(\frac{2.875}{3.375}\right)} = 62.37 \text{ s}$$

$$E = E_0 e^{-\frac{t}{\tau}} \Rightarrow 0.100 = 3.375 e^{-\frac{t}{\tau}}$$

$$\ln\left(\frac{0.100}{3.375}\right) = -\frac{t}{\tau} \Rightarrow t = -\tau \ln\left(\frac{0.100}{3.375}\right)$$

$$t = -62.37\ln\left(\frac{0.100}{3.375}\right) = 219.5 \text{ s}$$

(ii) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{300}{0.5}} = 24.5 \text{ rad/s}$

$$f = \frac{\omega}{2\pi} = \frac{24.5}{2\pi} = 3.90 \text{ Hz}$$

38. A 100-g object oscillates on a spring with a period of 5 s. Its first oscillation has an amplitude of 10 cm, and its sixth oscillation has an amplitude of 9 cm. Find (i) the time constant τ, (ii) the *Q*-factor. (iii) the fraction of the oscillator's energy dissipated each cycle, (iv) the amplitude and energy of the oscillator in its hundredth oscillation.

Solution:

(i) $A = A_o e^{-t/2\tau}$

It will take 5 periods (5*T*) to go from the first oscillation to the sixth oscillation.

$$9 = 10e^{-\frac{5T}{2\tau}} \Rightarrow \ln 0.9 = -\frac{5T}{2\tau}$$

$$\tau = \frac{-5T}{2\ln 0.9} = 23.73T = 23.73\times 5 \text{ s} = 118.6 \text{ s}$$

(ii) $Q = \omega\tau = \frac{2\pi\tau}{T} = \frac{2\pi\times 118.6}{5} = 149$

(iii) $\frac{\Delta E}{E} = \frac{2\pi}{Q} = \frac{2\pi}{149} = 0.042$

Or

$$\frac{\Delta E}{E} = \frac{T}{\tau} = \frac{5}{118.6} = 0.042$$

(iv) $A = A_o e^{-\frac{t}{2\tau}} = 10e^{-\frac{99T}{2\tau}} = 10e^{-\frac{99\times 5}{2\times 118.6}} = 1.24 \text{ cm}$

$$E = E_o e^{-\frac{t}{\tau}}$$

$$E_o = \tfrac{1}{2}kA_0^2$$

$$\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 m = \frac{4\pi^2 m}{T^2}$$

$$k == \frac{4\pi^2\times 0.1}{5^2} = 0.1579 \text{ N/m}$$

$$E_o = \tfrac{1}{2}(0.1579)(0.1)^2 = 7.895\times 10^{-4} \text{ J}$$

$$E = (7.895\times 10^{-4})e^{-\frac{99T}{\tau}} = (7.895\times 10^{-4})e^{-\frac{99\times 5}{118.6}} = 1.22\times 10^{-5} \text{ J}$$

39. The position of a particle is given by $x = 4\sin 2t$, where *x* is in meters and *t* is in seconds.

(i) What is the maximum value of *x*? What is the first time after $t = 0$ at which this maximum value occurs?

(ii) Find an expression for the velocity of the particle as a function of time. What is the velocity at $t=0$?
(iii) Find an expression for the acceleration of the particle as a function of time. What is the acceleration at $t=0$? What is the maximum value of the acceleration?

Solution:

(i) $A=4$ m

$x=4\sin 2t \Rightarrow 4=4\sin 2t$

$\sin 2t=1$

$$2t=\sin^{-1}1=\frac{\pi}{2}$$

$$t=\frac{\pi}{2}\text{s}$$

(ii) $v=\dfrac{dx}{dt}=8\cos 2t$

$v(0)=8\cos 0=8$ m/s

(iii) $a(t)=\dfrac{dv}{dt}=-16\sin 2t$

$a(0)=-16\sin 0=0$

$a_{\max}=-16\text{ m/s}^2$

40. When the displacement of a body oscillating on a spring is half its amplitude, what fraction of its total energy is its kinetic energy? At what displacement are its kinetic and potential energies equal?

Solution:

$K.E=\frac{1}{2}mv^2$

$v=\omega\sqrt{A^2-x^2}$

At $x=\dfrac{A}{2}$, $v=\omega\sqrt{A^2-\left(\dfrac{A}{2}\right)^2}=\omega\sqrt{A^2-\dfrac{A^2}{4}}=\omega\sqrt{\dfrac{4A^2-A^2}{4}}=\dfrac{\sqrt{3}}{2}\omega A$

$$K.E=\frac{1}{2}m\left(\frac{\sqrt{3}}{2}\omega A\right)^2=\frac{3}{4}\left(\frac{1}{2}kA^2\right)$$

$$K.E=\frac{3}{4}E_T \Rightarrow \frac{K.E}{E_T}=\frac{3}{4}$$

$K.E=P.E \Rightarrow \frac{1}{2}mv^2=\frac{1}{2}kx^2$

$mv^2=kx^2$

$v=\omega\sqrt{A^2-x^2}$

$$m\left(\omega\sqrt{A^2-x^2}\right)^2=kx^2$$

$m\omega^2\left(A^2-x^2\right)=kx^2 \Rightarrow k\left(A^2-x^2\right)=kx^2$

$A^2-x^2=x^2 \Rightarrow 2x^2=A^2$

$$x = \frac{A}{\sqrt{2}}$$

41. A mass of 1.53 kg is attached to a spring and the system is undergoing simple harmonic oscillations with a frequency of 1.95 Hz and an amplitude of 7.5 cm. What is the speed of the mass when it is 3.00 cm from its equilibrium position?

A. 0.0368 m/s B. 0.551 m/s C. 0.421 m/s D. 0.842 m/s E. 0.919 m/s

Solution:

Assuming $x(t) = A\cos\omega t$

$A = 7.5$ cm, $\omega = 2\pi f = 2\pi(1.95) = 12.257.5$ rad/s

$$3 = 7.5\cos 12.25t \Rightarrow t = \frac{\cos^{-1}(3/7.5)}{12.25}$$

$t = 0.095$ s

$$v(t) = \frac{dx(t)}{dt} = -\omega A\sin\omega t$$

$v(t) = -(12.25\times 7.5)\sin(12.25\times 0.095) = 84.2$ cm/s

Or

$$v = \omega\sqrt{A^2 - x^2} = 2\pi f\sqrt{A^2 - x^2} = 2\pi(1.95)\sqrt{0.075^2 - 0.03^2} = 0.842 \text{ m/s}$$

The correct option is D

42. If the displacement of a moving particle at any time t is given by the equation $x = 5\cos\omega t + 12\sin\omega t$. Calculate the amplitude of the motion.

Solution:

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)} = \sqrt{5^2 + 12^2} = 13$$